MagIC code : theory

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Fundamental equations

Conservation principles

1. **mass:** $\partial_t \rho + \nabla \cdot \left(\rho \mathbf{V}\right) = 0$

2. **momentum:** $\rho D_t \mathbf{V} = \underbrace{-\nabla P}_{\textbf{Pressure}} + \underbrace{\rho \mathbf{g}}_{\textbf{Buoyancy}} + \underbrace{\mathbf{F}^{\nu}}_{\textbf{Viscous force}}$

   - radial gravity $\mathbf{g} = g(r)\mathbf{e}_r$
   - viscous force $\mathbf{F}^{\nu} = \partial_j \tau_{ij}$ with $\tau_{ij} = \mu\left[\left(\partial_i v_j + \partial_j v_i\right) - 2/3 \partial_k v_k \delta_{ij}\right]$
     $\mu =$ dynamic viscosity $[\mathrm{M}\,\mathrm{L}^{-1}\,\mathrm{T}^{-1}]$

3. **energy:** $\rho T D_t S + \nabla \cdot (\underbrace{\mathbf{I}^{q}}_{\textbf{Heat flux}}) = \underbrace{Q^{\nu}}_{\textbf{Viscous heating}}$

   - Viscous heating $Q^{\nu} = \tau_{ij} \nabla_j V_i$

- + **equation of state** $f(\rho, P, T) = 0$ ex. perfect gas $P = R\rho T$

Why developping convective approximations ?

Twofold objective: physical and numerical

1. retain the essential physics with a minimum complexity
2. filter out sound waves : low Mach number approximation

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The Boussinesq Approximation

$$\nabla \cdot (\mathbf{u}) = 0$$

- J. Boussinesq (1872), Rayleigh (1916)
- Numerical benchmark : Christensen et al. 2001

The Anelastic Approximation

$$\nabla \cdot \left(\overline{\rho_\mathrm{a}}\, \mathbf{u}\right) = 0$$

- Spiegel and Veronis (1960), Gilman (1980)
- Numerical benchmark : Jones et al. 2011

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The reference state

Idea

from a thermodynamics point of view, consider the convective state as a small departure from a steady *reference state*

- $\rho = \rho_a + \rho_c$
- $T = T_a + T_c, \ldots$
- $\frac{x_c}{x_a} \sim \epsilon \ll 1$

The reference state must be in quasiequilibrium:

Mechanical quasiequilibrium

hydrostatic balance

$$-\nabla P_a + \rho_a \mathbf{g} = 0$$

Thermal quasiequilibrium

“well mixed” state

$$\nabla S_a = 0$$

This leads to the adiabatic temperature gradient (Schwarzschild criterion)

$$\nabla T_a = \left(\frac{\partial T}{\partial P}\right)_S \nabla P_a = \frac{\alpha T_a}{C_p}\mathbf{g} = \frac{\mathbf{g}}{C_p} \quad \text{(for a perfect gas)}$$

Example of reference states

Polytropic

when $g \propto 1/r^2$, there is an analytic solution

$$P_a = P_r \zeta(r)^{n+1}$$

$$\rho_a = \rho_r \zeta(r)^n$$

$$T_a = T_r \zeta(r)$$

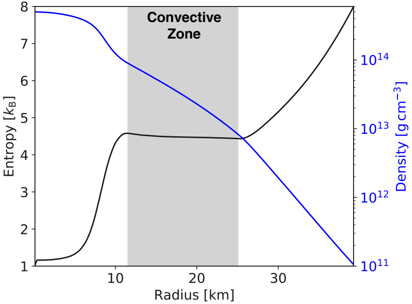
Control parameters:

1. polytropic index n
2. density stratification $N_\rho = \ln\left(\frac{\rho_{\text{bottom}}}{\rho_{\text{top}}}\right)$

More realistic

Fit $(\tilde{\rho}, \tilde{T})$ of any 1D interior model

- Jupiter
- proto-neutron star, ...



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Treatment of the buoyancy term

Idea: $\rho' = f(s, P')$

$$\varrho' = \left(\frac{\partial \varrho_a}{\partial s}\right)_p s + \left(\frac{\partial \varrho_a}{\partial p}\right)_s P' = -\frac{\varrho_a}{c_p} s - \frac{1}{\varrho_a g}\frac{\partial \varrho_a}{\partial z} P'$$

Simplification :

$$\frac{1}{\rho_a}\nabla P' + \frac{\rho'}{\rho_a}\mathbf{g} = -\nabla\left(\frac{P'}{\rho_a}\right) - \frac{s}{c_p}\mathbf{g}$$

Lantz-Braginsky-Roberts formulation

$$\nabla \cdot (\varrho_a \mathbf{u}) = 0$$

$$\frac{D\mathbf{u}}{Dt} = -\nabla\left(\frac{P'}{\varrho_a}\right) - \frac{s}{c_p}\mathbf{g} + \frac{\mathbf{F}^v}{\varrho_a}$$

$$\varrho_a T_a \frac{Ds}{Dt} = -\nabla \cdot \mathbf{I}^q + Q^v$$

Problem: $\mathbf{I}^q = -k\nabla T$? (Fourier)

Treatment of the heat flux: a closure problem

Solve the equation for average quantities

$$f = \langle f \rangle + f^t \tag{1}$$

This creates a turbulent entropy flux term

$$\mathbf{I}^{St} = \rho_a \langle S^t \mathbf{V}^t \rangle \propto \nabla \langle S \rangle \tag{2}$$

Strong simplification

- the usual term $\mathbf{I}^q = -k\nabla T$ is then neglected in the heat transfer equation
- $\Longrightarrow$ **temperature is removed from the problem !**
- energy budget ok (problem with other formulations...)

Drawback

- P2 of thermodynamics not under warranty anymore

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Derivation of the Boussinesq approximation (BA)

Idea

- thin layer approximation $\eta = \frac{gd}{C_P T_r} = \frac{d}{H_{\text{hydrostatic}}} \ll 1$
- neglect pressure perturbation $\rho = \rho_0(1 - \alpha(T - T_0))$

Resulting system

$$\nabla \cdot (\mathbf{u}) = 0$$

$$D_t \mathbf{u} = -\nabla \Pi - \alpha T \mathbf{g} + \nu \nabla^{\mathbf{2}} \mathbf{u}$$

$$D_t T = \kappa \nabla^{\mathbf{2}} T$$

Remarks

- Fourier's law is back, viscous heating negligible
- consistent with $N_\rho \to 0$ (polytropic reference state)

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The MHD anelastic equations

Braginsky+95, Lantz+99, Jones+[11,14]

$[d] = r_o - r_i,\quad [t] = d^2/\nu_o,\quad [S] = d\,\partial S/\partial r|_{r_o},\quad [p] = \Omega\varrho_o\nu_o,\quad [B] = \sqrt{\Omega\varrho_o\mu_0\eta_o}$

$$\nabla\cdot\mathbf{B} = 0$$

$$\frac{\partial\mathbf{B}}{\partial t} = \underbrace{\nabla\times(\mathbf{u}\times\mathbf{B})}_{\textbf{Induction}} - \underbrace{\frac{1}{Pm}\nabla\times\left(\eta\nabla\times\mathbf{B}\right)}_{\textbf{Dissipation}}$$

$$0 = \nabla\cdot\left(\tilde{\varrho}\mathbf{u}\right)$$

$$\frac{D\mathbf{u}}{Dt} = -\overbrace{\nabla\left(\frac{p}{E\tilde{\varrho}}\right)}^{\textbf{Pressure}} - \overbrace{\frac{2}{E}\mathbf{e}_z\times\mathbf{u}}^{\textbf{Coriolis}} - \overbrace{\frac{Ra}{Pr}\frac{d\tilde{T}}{dr}}^{\textbf{Buoyancy}}S\mathbf{e}_r + \overbrace{\mathbf{F}_\nu}^{\textbf{Viscosity}} + \overbrace{\frac{1}{EPm}\frac{1}{\tilde{\varrho}}(\nabla\times\mathbf{B})\times\mathbf{B}}^{\textbf{Lorentz}}$$

$$\frac{DS}{Dt} = \frac{1}{Pr\tilde{\varrho}\tilde{T}}\underbrace{\nabla\cdot\left(\kappa\tilde{\varrho}\tilde{T}\nabla S\right)}_{\textbf{Heat flux}} + \frac{Pr}{Ra\tilde{\varrho}\tilde{T}}\left(\underbrace{\frac{\eta}{Pm^2E}(\nabla\times\mathbf{B})^2}_{\textbf{Ohmic heating}} + \underbrace{Q_\nu}_{\textbf{Viscous heating}}\right)$$

Dimensionless control parameters

Rayleigh number	Ra	$\frac{\alpha_o T_o g_o d^3 \Delta S}{c_p \nu \kappa}$	$\frac{\text{buoyancy}}{\text{viscous effects}}$
Ekman number	E	$\frac{\nu}{\Omega d^2}$	$\frac{\text{viscosity}}{\text{Coriolis}}$
Prandtl number	Pr	$\frac{\nu}{\kappa}$	$\frac{\text{viscosity}}{\text{thermal diffusivity}}$
magnetic Prandtl number	Pm	$\frac{\nu}{\eta}$	$\frac{\text{viscosity}}{\text{magneticdiffusivity}}$

Onset of convection

Convection occurs a certain critical value of the Rayleigh number

$$Ra_c = f(E, \text{adiabatic reference state}, \text{geometry})$$

Physical parameter regime and numerical limitations

Parameter	Earth's core	Gas giants	Sun	Simulations
E	10^{-15}	10^{-18}	10^{-15}	10^{-6}
Ra	10^{27}	10^{30}	10^{24}	10^{12}
Pr	10^{-1}	10^{-1}	10^{-6}	10^{-1}
Pm	10^{-6}	10^{-7}	10^{-3}	10^{-1}
Re	10^{9}	10^{12}	10^{13}	10^{3}

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Conclusion

- AA and BA : turbulent fluctuations from an adiabatic state
- low Mach flow $\rho'/\rho_a = O(M^2)$

Boussinesq

- valid for “smal” system ($d \ll H_p$)
- consider usually constant fluid parameters (viscosity, thermal diffusivity)

Anelastic

- proper way to take into account the radial stratification
- similar mathematical structure

But…

- be careful to the formulation : prefer the LBR one