Introduction to Space Plasma physics

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Chapter 1: Ionization and recombination*

This chapter will not seen during the course and is not at the program of the exam

From planetary environments to the intergalactic medium, most of the visible matter in the universe is ionized. The main reason for this is the existence of stars, which are sources of ionizing radiations.

Plasma state results from the balance between two competing processes : ionization, which is the production of a positive ion and a free electron from a neutral atom, and recombination, which is the inverse process. The two major ways to ionize an atom are by electron impact, or by photo-ionization. In this chapter, we briefly describe how to quantify these effects.

1.1 Thermal equilibrium

In thermal equilibrium, the ionization degree of a gas resulting from the balance between all ionizing processes (collisions, radiation...) is given by Saha's equation. It is important to note that since the equilibrium involves the degrees of freedom of the electromagnetic radiation, this equation is valid only in an opaque (optically thick) medium.

$$\frac{n_e n_{X^{n+1}}}{n_{X^n}} = \frac{2g_{X^{n+1}}}{g_{X^n}} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-W/kT} \tag{1.1}$$

n_e is the free electron density, X^n and X^{n+1} are two consecutive ionization ground states of the atomic specie X. W is the potential of ionization, i.e. the difference of energy between the ground states X^n and X^{n+1}.

The result of this equation in the case of a gas of hydrogen, for different densities of nuclei $n_{tot} = n_{H^+} + n_H$ is illustrated on Fig.1.1. We see that the ionization degree is high even for temperature quite smaller than the ionization energy. This illustrates that, at low densities, recombination is a rather inefficient process.

The mass density of the photosphere of the Sun (which is an optically thick medium in which Saha's law is applicable) is around 3×10^{-4} kg.m^{-3}, which corresponds to a

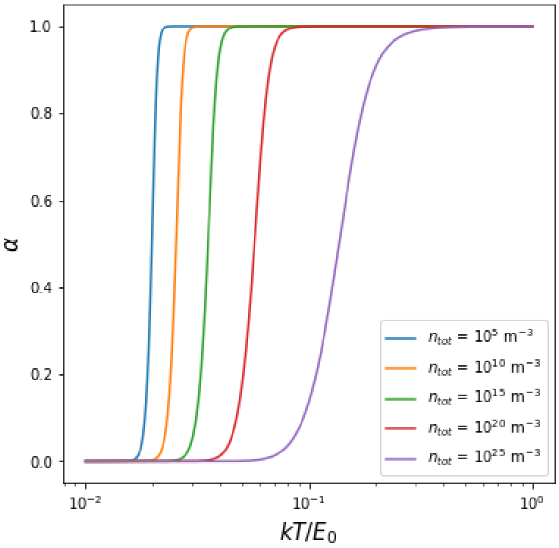

FIGURE 1.1 – Ionisation degree $\alpha = n_e/(n_e + n_{H^+})$ as a function of the temperature for different densities.

number density of protons of $n = 2 \times 10^{23}$ m^{-3}. The observed temperature is $T = 6400$ K. Saha's equation gives a value of $\alpha \simeq 4 \times 10^{-4}$: the photosphere is almost neutral.

In the atmospheric layers above the photosphere, which are (by definition of the photosphere...) optically thin, Saha's equation is not valid. This happens to be the case of most dilute plasmas, which makes Saha's equation of little concrete use in plasma physics, outside of stellar interriors.

1.2 Out of equilibrium description

The out of equilibrium description of the ionization degree of a plasma passes by the description of each possible ionization and recombination reaction. The efficiency of a process is characterized by its cross-section, on which we give a reminder below.

1.2.1 Reaction cross-section and mean-free path

Consider the reaction

$$A + B \to C + D. \tag{1.2}$$

Consider a volume containing "target" particles B at rest. A flux density $\Phi_A = n_A v_{A/B}$ of particles A is passing through the volume (n_A and $v_{A/B}$ are the number density and velocity of the particles A with respect to particles B). The number of reaction per unit volume and unit time is

$$\dot{n}_{reac} = dn_C/dt = -dn_A/dt = n_A n_B v_{A/B} \sigma \tag{1.3}$$

which defines the cross section σ, homogeneous to a surface. Of course the equation is unchanged by exchanging indices A and B ($v_{A/B} = v_{B/A}$ is here a positive value).

We can also define the mean-free path λ of the specie A "colliding" with the background made of specie B from the small probability dp that A interacts with B while travelling a small distance ds : $\lambda^{-1} = dp/ds$. Then we have

$$n_A(s + ds) = n_A(s)(1 - dp) \Rightarrow dn_A/ds = -n_A/\lambda \tag{1.4}$$

from the definition of λ. So, if a beam of particles A is launched in a medium characterized by a constant mean-free path, its density will decrease exponentially along its way, with a characteristic decay length equal to λ.

Now making the connection with the cross section : we simply have the distance travelled $ds = v_{A/B}dt$, so the number of reaction per unit time and volume is, from the previous equation

$$dn_A/dt = -n_A v_{A/B}/\lambda \Rightarrow \lambda = (n_B \sigma)^{-1} \tag{1.5}$$

1.2.2 Ionization processes

The two main ionization reactions are the ionization by electron impact and the photo-ionization. The first correspond to the equation

$$A + e^- \rightarrow A^+ + e^- + e^- \tag{1.6}$$

and is characterized by a cross section σ_e given, in the classical approximation, by the Thomson formula [1]

$$\sigma_e(E) = \frac{\pi q_e^4 (E - W)}{W E^2} \tag{1.7}$$

where E is the ionizing electron's energy in the rest frame of the impacted atom, and $q_e \equiv e/\sqrt{4\pi\varepsilon_0}$. The maximum of this function is reached at twice the ionisation potential W, and is $\sigma_e \sim 10^{-20}$ m^2 for $W \sim 10$ eV.

The photoionisation reaction involves the interaction with a photon,

$$A + h\nu \rightarrow A^+ + e^-. \tag{1.8}$$

It is characterized by a cross section σ_{ph} that is a function of the energy $h\nu$ of the photon and of the chemical properties of A. There is no simple, classical expression for this cross section, but a good order of magnitude, in practical cases, is given by

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[1] A demonstration of this formula will be given in the chapter on collisions.

$\sigma_{ph} \simeq 5 \times 10^{-22}(E/W)^{-3}$ m^2, for photo-electrons having energies larger than the ionisation potential.

The ionising source in most astrophysical cases will be a star. In this case the flux density of ionising photons at a distance r from the star will be related to the star spectral luminosity L_ν by

$$N_{ph} = \frac{1}{4\pi r^2}\int_{\nu_0}^{\infty}\frac{L_\nu}{h\nu}d\nu \tag{1.9}$$

where $\nu_0 = W/h$ is the minimal ionisation frequency, and, approximating the star radiation by a perfect blackbody,

$$L_\nu(R,T) = 4\pi R^2\frac{2\pi h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} \tag{1.10}$$

where R and T are the star radius and temperature, respectively. The typical number of ionizing photons (i.e. $h\nu \gtrsim 10$ eV) for the Sun is $4\pi r^2 N_{ph} \sim 1.8 \times 10^{33}$ photons.s^{-1}. For a B type main sequence star ($T = 30000$ K), $4\pi r^2 N_{ph} \sim 1.5 \times 10^{48}$ photons.s^{-1}.

1.2.3 Recombination processes

Recombination is the inverse process of ionisation. The ionisation reactions seen above can be read in the opposite direction, giving the three-body recombination

$$A^+ + e^- + e^- \rightarrow A + e^- \tag{1.11}$$

which is in most dilute plasma cases a rather inefficient process.

The inverse of the photoionisation is the radiative recombination,

$$A^+ + e^- \rightarrow A + h\nu \tag{1.12}$$

which is an important process in astrophysics. It is usually, in the absence of molecular species, the dominent process, and is characterized by a cross section $\sigma_{rad} \sim 10^{-24}$m^2. Note that it is much smaller than the ionisation cross section.

In the presence of diatomic molecules, there exist a much more efficient recombination process, which is the dissociative recombination

$$AB^+ + e^- \rightarrow A^* + B^* \tag{1.13}$$

Its cross section is $\sigma_{dis} \sim 10^{-18}$ m^{-2} : this process happens to be dominent in most regions of the Earth's ionosphere, which is mainly constituted of N_2 or O_2 molecules.

1.3 An example : Chapman's ionization layer

An example of a space plasma (mainly) produced by photo-ionization is the Earth's ionosphere. Here we present a classical historical model for the structure of the ionospheric plasma layer.

We assume that the atmosphere is composed by a single chemical specie of molecular mass m, in hydrostatic equilibrium at temperature T with a base density n_0, so that the molecular number density is

$$n(z) = n_0 e^{-z/H}, \qquad H = \frac{kT}{mg} \tag{1.14}$$

g is the acceleration of gravity, assumed constant. We assume this exponential atmosphere is impacted by solar ionizing radiation coming along the direction of the z axis, and characterized by a photon flux density $N_{ph}(z)$ in $\mathrm{m}^{-2}\mathrm{s}^{-1}$.

The probability that a photon gets absorbed when it travels a distance dz is $dp = dz/\lambda_{ph} = n(z)\sigma_{ph} dz$. Therefore the flux of photons evolves with z according to

$$N_{ph}(z - dz) = N_{ph}(z)(1 - dp) \Rightarrow \frac{dN_{ph}}{dz} = n(z)\sigma_{ph} N_{ph} \tag{1.15}$$

from which we obtain

$$N_{ph}(z) = N_{ph}(\infty) \exp\left(-\sigma_{ph} \int_z^\infty n(z)dz\right) \tag{1.16}$$

where we have assumed $n(z)$ is a given function (i.e. the ionization process does produce an ion/electron density that is small compared to the neutral density). The argument of the exponential is called the optical depth of the atmosphere at altitude z. $N_{ph}(\infty)$ is the ionizing photon flux from the Sun at the Earth's orbit.

We can calculate $N_{ph}(z)$ using eq.(1.14) :

$$N_{ph}(z) = N_{ph}(\infty) \exp\left(-\sigma_{ph} n(z) H\right) \tag{1.17}$$

which gives the profile of the Sun's UV light in the atmosphere. According to eq.(1.8), an electron is created each time a photon is absorbed. So, the number electron created per unit volume checks $Q_e dz = N_{ph}(z + dz) - N_{ph}(z)$, and $Q_e = dN_{ph}/dz$. We can evaluate this quantity using the previous results,

$$Q_e = -n(z)\sigma_{ph} N_{ph}(z) = N_{ph}(\infty)\sigma_{ph} n_0 \exp\left(-\sigma_{ph} n(z) H - z/H\right) \tag{1.18}$$

that is often put into the form

$$Q_e = Q_{max} \exp\left(1 - y - \exp -y\right) \tag{1.19}$$

and is called the Chapman's electron production function, from the British plasma physicist Sydney Chapman.

The number density of electrons in the ionosphere is now obtained by assuming steady-state equilibrium between electron production and recombination. The dissociative recombination is dominent, and occurs with a volumic rate R, according to eq(1.13),

$$R = k_{dis} n_e^2, \qquad k_{dis} = \sigma_{dis} v_e \tag{1.20}$$

v_e being the thermal speed of the free electrons. The steady-state $R = Q_e$ gives the electron number density in the ionosphere

$$n_e(z) = \sqrt{Q_e(z)/k_{dis}} \tag{1.21}$$

Chapter 2: Collective phenomena and characteristic scales

Plasmas are fully or partially ionised gases. Such states are frequently found in astrophysical situations, due to the presence of stars, which are intense sources of ionising radiation : all matter in the vicinity of a star (as well as the star itself) is in the form of plasma – at larger scales with the exception of particularly protected environments, such as the Earth's surface (for which ionising radiation is mostly absorbed in the ionospheric layers of the atmsophere, cf. previous chapter, Chapman's layer).

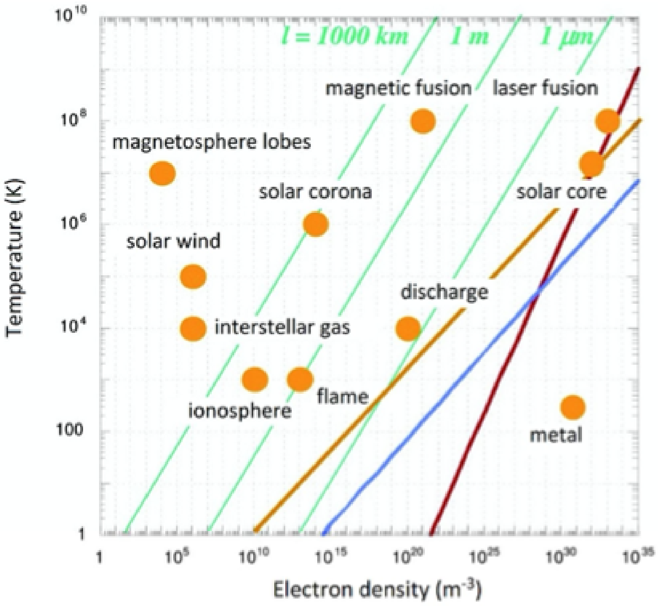

FIGURE 2.1 – Ionised environments in the density-temperature plane. The green lines indicate iso-contours of (coulomb-collisional) mean free path. The other three lines indicate plasmas for which the mean interparticle distance $d = n^{1/3}$ is equal to the Landau length (blue), the Debye length (yellow) and the de Broglie length (red).

Astrophysical plasmas can be characterised by a very wide range of parameters, as illustrated in Figure 2.1 : densities range from around 1 cm^{-3} in the Earth's magnetosphere and less (intergalactic medium...) to 10^{25} cm^{-3} inside the Sun. The table below presents some parameters relevant to various astrophysical environments.

Table 1: An overview of parameters of ionised gases in the Universe (adapted from J.A. Irwin, Astrophysics - Decoding the Cosmos, Wiley 2007)

Location	n_e [m^{-3}]	$n_m + n_H$ [m^{-3}]	T [K]	Reference
Solar corona	10^{14}	$\simeq 0$	10^6	Koutchmy 1994, Adv. Space Res. 14(4), 29
Interplanetary space (1 AU)	$10^6 - 10^8$	$\simeq 0$	$10^4 - 10^5$	Schwenn 1991, dans "Physics of the Inner Heliosphere", Springer
Stellar interiors	10^{33}	$\simeq 0$	$10^{7.5}$	
Planetary nebulae	$10^9 - 10^{11}$	$\simeq 0$	10^4	Zhang et al. 2004, MNRAS 351, 935
H II regions	$10^8 - 10^9$	$\simeq 0$	$10^3 - 10^4$	Feng-Yao Zhu et al 2019, ApJ 881, 14
Interstellar medium	$10^3 - 10^7$	$10^4 - 10^{11}$	10^2	compare with lecture on the ISM
Intergalactic space	< 10	$\simeq 0$	$10^5 - 10^6$	

number densities: n_e free electrons, n_m molecules, n_H neutral hydrogen

The dynamics of neutral gases are controlled by short-range interactions between their constituents, i.e. essentially by *binary collisions*. The situation is different in plasma physics, due to the effect of the Lorentz force, which causes long-range macroscopic correlations in plasma. This effect is intrinsically non-linear and involves feedback effects between the field and the ionised matter.

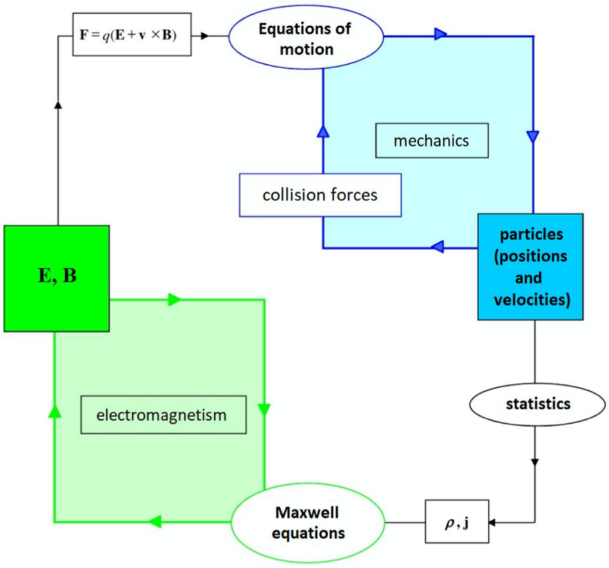

FIGURE 2.2 – Illustration of the field-matter feedback loop characteristic of plasma physics.

Figure 2.2 illustrates this feedback loop : the distributions of particles in phase space determine the charge and current densities, which are the sources of the electromagnetic field. The electromagnetic field, in turn, causes the particles to move through the Lorentz force, and thus to evolve in phase space.

Although this approach is easy to understand conceptually, it is generally difficult to implement due to the extremely high number of particles and the non-linearity of the equations linking dynamics and field.

To gain a practical understanding, it is therefore unreasonable to attempt to consider the problem in all its complexity. Instead, we will study plasmas by considering specific phenomena. For each phenomenon considered, appropriate simplifying approximations are made, leading to an easier problem to deal with, isolating a particular physical process. A situation in which a certain set of approximations is valid and provides a consistent description is called a *regime.*

2.1 Characteristic scales : a first approach

In a neutral gas of point particles of mass m, characterised by its density n and temperature T, two independent scales can be constructed : the interparticle distance $\ell = n^{-1/3}$, and the thermal velocity, or (isothermal) speed of sound, $v_{th} = c_s = \sqrt{kT/m}$.

If we consider a gas of charged particles, we must add the typical charge of a particle e and the constant ε_0. Let's do a quick dimensional analysis[1] and look for a length scale in the form $(\epsilon_0/e^2)^\alpha (m)^\beta n^\gamma (kT)^\delta$. We get the system

$$\begin{cases} -\alpha + \beta + \delta = 0 \\ -3\alpha - 3\gamma + 2\delta = 1 \\ 2\alpha - 2\delta = 0 \end{cases} \Rightarrow \begin{cases} \alpha = \delta \\ \beta = 0 \\ \gamma = -(\alpha+1)/3 \end{cases}$$

We have 4 unknowns and 3 equations, so we can a priori construct an infinite number of lengths... Interesting special cases are

- $\alpha = 0$ gives $\ell = n^{-1/3}$: the interparticle distance.
- $\alpha = -1/2$ gives $\lambda_D = (\varepsilon_0 kT/ne^2)^{1/2}$ the Debye length, or electrostatic screening length in a plasma.
- $\alpha = -1$ gives $\lambda_L = e^2/(\varepsilon_0 kT)$, the Landau length, which is (to within a factor of 4π...) the distance separating two particles with thermal energies equal to their pair electrostatic interaction energy.

Note that these lengths are not independent, since

$$\lambda_D^2 \lambda_L \sim \ell^3 \tag{2.1}$$

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[1] $[\epsilon_0/e^2] = M^{-1}L^{-3}T^2$, $[n] = L^{-3}$ and $[kT] = ML^2T^{-2}$

Using the thermal velocity, we can construct a characteristic frequency,

- $\omega_p = v_{th}/\lambda_D = (ne^2/m\varepsilon_0)^{1/2}$ which is the plasma frequency ; this is, as we will see, the typical frequency of oscillation of a charge density perturbation in a plasma.

Finally, if we consider a magnetised plasma, characterised by its magnetic field B and the constant μ_0, other characteristic scales associated with the dynamics of individual particles appear :

- $\omega_c = eB/m$ is the frequency of rotation of a particle around a magnetic field line.
- $\rho = v_{th}/\omega_c$ is the thermal Larmor radius (magnetic rotation radius).

But also new "collective" scales (involving density n) :

- $V_A = (B^2/nm\mu_0)^{1/2}$ is the Alfvén speed, which is the propagation speed of a transverse magnetic field perturbation in a plasma.
- $\lambda = c/\omega_p \equiv 1/\sqrt{\varepsilon_0\mu_0}\omega_p$ the so-called inertial length.

It should therefore be noted that, unlike a neutral gas, a plasma, particularly a magnetised plasma, is characterised by a very large number of characteristic scales — and therefore by a wide variety of regimes and physical phenomena. In the following, we will discuss the physical interpretation of (some of) these scales and the description of some of these phenomena.

2.2 Debye screening and electro-neutrality : the quasi-stationary regime

As a medium rich in free electrons, a plasma is a good electrical conductor. And we know that a conductor in equilibrium cannot maintain a non-zero charge density in its volume. We therefore expect that a plasma in a steady state will contain neither charge density nor macroscopic electric field. If we introduce a charged object into a plasma, the electrons and ions will distribute themselves in space in such a way as to cancel out the electric field produced by this charge. Here we see how.

2.2.1 Electrostatic screening

We assume that each species s in the plasma is described by its density n_s, its average velocity $\mathbf{u}_s$ and its pressure $p_s = n_s k T_s$. The conservation of the momentum density of population s is described by the equation

$$n_s m_s \frac{d\mathbf{u}_s}{dt} = n_s q_s \mathbf{E} - \nabla p_s \tag{2.2}$$

where we neglect any friction between particle populations (we will return to this assumption later).

Let's investigate the *quasi-static regime*, $d/dt \to 0$ (we are therefore describing a *slow* phenomenon, relative to a characteristic time that we will determine later). It can be noted that this amounts to neglecting the inertia of the particles, $m_s \to 0$. We also assume that the electric field is purely potential, $\mathbf{E} = -\nabla\varphi$. The previous equation reduces to

$$0 = -n_s q_s \nabla\varphi - \nabla p_s. \tag{2.3}$$

We assume isothermal populations : $T_s = const.$, and therefore

$$n_s = n_{0,s} \exp\left(-\frac{q_s\varphi}{kT_s}\right). \tag{2.4}$$

In the quasi-static regime, the spatial distributions of charged particles are therefore given by Boltzmann's law.

Let us now study the effect of a point test particle with charge q_t inserted into a plasma that is otherwise globally neutral and spatially homogeneous. The electric potential in the plasma is described by Poisson's equation

$$\Delta\varphi = -\frac{\rho_{pol}(\mathbf{r})}{\varepsilon_0} - \frac{q_t\delta(\mathbf{r})}{\varepsilon_0} \tag{2.5}$$

where we assume that the test charge is placed at the origin of the coordinate system. The charge density ρ_{pol} describes the polarisation of the plasma charges in the electric field created by the test charge. It is given by

$$\rho_{pol}(\mathbf{r}) = \sum_s q_s n_s(\mathbf{r}) = \sum_s q_s n_{0,s} \exp\left(-\frac{q_s\varphi}{kT_s}\right) \tag{2.6}$$

We assume that the disturbance produced by the test particle is of small amplitude, so that the potential energy $q_s\varphi$ of the particles is small compared to their thermal energy – except perhaps in the immediate vicinity of the test charge. We can therefore expand

$$n_s(\mathbf{r}) = n_{0,s}\left(1 - \frac{q_s\varphi}{kT_s} + \frac{1}{2}\left(\frac{q_s\varphi}{kT_s}\right)^2 + ...\right), \tag{2.7}$$

from which we obtain, to first order

$$\rho_{pol}(\mathbf{r}) \simeq -\sum_s \frac{n_{0,s}q_s^2\varphi}{kT_s}, \tag{2.8}$$

where we have assumed that the plasma as a whole (apart from the test charge) is neutral, $\sum_s n_{0,s}q_s = 0$.

Poisson's equation (2.5) can now be written in linearised form

$$\Delta\varphi - \frac{\varphi}{\lambda_D^2} = -\frac{q_t\delta(\mathbf{r})}{\varepsilon_0} \tag{2.9}$$

where we have introduced the characteristic length, known as the Debye length,

$$\lambda_D^{-2} = \sum_s \frac{q_s^2 n_{0,s}}{\epsilon_0 k T_s} \equiv \sum_s \lambda_{D,s}^{-2} \tag{2.10}$$

The solution to eq.(2.9) (known as the Yukawa equation) is

$$\varphi(r) = \frac{q_t}{4\pi\epsilon_0 r} \exp(-r/\lambda_D) \tag{2.11}$$

We can see that the potential near the test charge is its potential in a vacuum, and that the potential decreases exponentially (this is the screening effect) from distances of the order of the Debye length.

Numerically, the electron Debye length (or any species carrying a charge e...) is :

$$\lambda_{De}\ [\mathrm{m}] \simeq 7.4 \left(\frac{T_e\ [\mathrm{eV}]}{n_e\ [\mathrm{cm}^{-3}]} \right)^{1/2} \tag{2.12}$$

We made several assumptions to obtain this result : that the plasma was non-collisional (no friction term), that the potential perturbation due to the test charge was small compared to the thermal energy of the plasma particles, and that the perturbation was *slow*. Regarding this last assumption, we can now estimate, in order of magnitude, the inertial term on the left side of eq.(2.2) : if we consider a perturbation acting at a typical frequency ω, the inertial term will be negligible if

$$m\omega u \ll kT/\lambda_D \iff \omega \ll v_{th}^2/u\lambda_D. \tag{2.13}$$

If we assume a subsonic population ($u \ll v_{th}$), a sufficient condition for the validity of the quasi-static hypothesis is therefore $\omega \ll \omega_p$, where the *plasma frequency* is defined as

$$\omega_p = v_{th}/\lambda_D = \sqrt{\frac{ne^2}{m\varepsilon_0}}. \tag{2.14}$$

This frequency describes the speed of the plasma's inertial response to a disturbance. Any disturbance that is very slow compared to $\omega_{p,s}$ will be screened by species s.

2.2.2 The plasma parameter and the hierarchy of spatial scales

We have assumed that the potential disturbance due to the test charge is small compared to the thermal energy of the plasma particles. We now have an expression for the potential φ in the plasma. The inter-particle distance in the plasma is $\ell = n^{-1/3}$. We have

$$\frac{e\varphi(\ell)}{kT} \simeq \frac{\Gamma^{2/3}}{4\pi} \exp(-\Gamma^{1/3}) \tag{2.15}$$

where we have introduced the *plasma parameter*

$$\Gamma = \frac{1}{n\lambda_D^3}, \tag{2.16}$$

which is therefore the inverse of the average number of particles in a "Debye cube". The assumption of a low potential value compared to thermal energy is valid if $\Gamma \ll 1$, which means that a typical volume λ_D^3 must contain a large number of particles.

This condition is necessary for a set of charged particles to exhibit screening properties and therefore behave like a plasma. A plasma therefore always satisfies, by definition, $\Gamma \ll 1$. If this condition is not met, we refer to a "Coulombian gas" rather than a plasma. This condition is always very well satisfied in ionised astrophysical environments. For example, in the solar wind at 1 AU, $\lambda_D \simeq 10$ m and $n \simeq 5$ cm^{-3}, hence $\Gamma \simeq 2 \times 10^{-10}$.

The smallness of Γ makes it possible to order the previously introduced plasma distance scales ℓ, λ_L and λ_D. We have $\ell/\lambda_D = \Gamma^{1/3} \ll 1$, and $\ell/\lambda_L = (\lambda_D/\ell)^2 = \Gamma^{-2/3} \gg 1$. The spatial scales of the plasma are therefore ordered as follows :

$$\lambda_L \ll \ell \ll \lambda_D, \tag{2.17}$$

with ratios controlled by the value of the plasma parameter Γ.

2.2.3 Electro-neutrality

Thermal fluctuation

A plasma, in the quasi-static approximation, is macroscopically *quasi-neutral* at scales larger than the Debye length. This can be understood intuitively by the calculation in paragraph 2.2.1, since the charge density ρ decreases exponentially from the vicinity of a test charge over a length λ_D. This can also be illustrated by calculating the maximum scale of a thermal fluctuation in charge density in a plasma.

Consider an initially neutral plasma with density n and temperature T, composed of ions with charge e and electrons. We seek to calculate the largest radius r_{max} of a sphere that could spontaneously empty itself of electrons due to thermal fluctuations. Let us assume that the thermal motion of the electrons is purely radial – thus the problem is spherically symmetric [2].

The electric field in the electron-empty sphere is purely radial, and given by Gauss's theorem,

$$E_r(r) = \frac{\rho_i r}{3\varepsilon_0} = \frac{enr}{3\varepsilon_0} \tag{2.18}$$

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[2] This situation is of course extremely artificial (in reality, the electrons would move in random directions and we would have to consider the distribution of radial velocities), but it is a "worst case" scenario, which allows us to give an upper bound on the size of the electron-empty sphere.

From this we can deduce the electrostatic energy associated with the electron density fluctuation,

$$W_E = \iiint \frac{\varepsilon_0 E_r^2}{2} dV = \int_0^{r_{max}} \frac{e^2 n^2 r^2}{18\varepsilon_0} 4\pi r^2 dr = \frac{2\pi n^2 e^2}{45\varepsilon_0} r_{max}^5 \tag{2.19}$$

This energy is provided by the thermal motion of electrons, so we have $W_E = W_{th}$, where the thermal energy of the electrons initially contained in the sphere is

$$W_{th} = \iiint \frac{3}{2} nkT dV = 2nkT\pi r_{max}^3 \tag{2.20}$$

From this we can deduce the size of our spherical fluctuation

$$r_{max}^2 = 45 \frac{\varepsilon_0 kT}{ne^2} = 45\lambda_{De}^2 \tag{2.21}$$

This calculation, which is obviously crude, shows that the largest volume that can spontaneously charge under the effect of thermal fluctuations has a radius of a few Debye lengths – in the highly improbable situation where all electrons initially move in the radial direction outwards. We can conclude that the plasma is quasi-neutral over length scales much larger than the Debye length.

Plasma approximation and ambipolar electric field

In the quasi-static regime, and on scales larger than λ_D, we can therefore state that, to a good approximation, the plasma is neutral :

$$\sum_s q_s n_s(\mathbf{r}) = 0. \tag{2.22}$$

This relationship is commonly referred to in the literature as the *plasma approximation.* It is a particularly useful approximation in that it allows the (potential) electric field in the plasma to be calculated without using Maxwell's equations. In a sense, it replaces the Maxwell-Gauss equation.

It should be noted that this macroscopic quasi-neutrality does not imply that the electric field is zero in the plasma : on the contrary, an electric field must generally exist in order to maintain electro-neutrality against other effects, and in particular inertial effects that would tend to induce charge separation between particles having different masses. Such a field is called an *ambipolar electric field.*

To illustrate this effect, let us consider a plasma in a steady state in an acceleration field $\mathbf{g}(\mathbf{r})$. If we naively assume that the electric field is zero in the plasma, we quickly arrive at a contradiction : the difference in mass between ions and electrons induces a difference in the density of these populations, so a space charge must exist, and therefore an electric field... The equation describing the equilibrium must therefore take into account an electric field :

$$0 = -kT_s \nabla n_s + n_s q_s \mathbf{E}(\mathbf{r}) + n_s m_s \mathbf{g}(\mathbf{r}) \tag{2.23}$$

We could solve two equations, for ions and electrons, coupled with Maxwell-Gauss (or Poisson) equation, to obtain the electric field : this would be quite cumbersome. Instead, we can assume the plasma approximation $n_e = n_i \equiv n_{plasma}$. Subtracting the equations for these two populations gives us the ambipolar electric field

$$\mathbf{E}(\mathbf{r}) \simeq -\frac{m_i \mathbf{g}(\mathbf{r})}{2e}. \tag{2.24}$$

and adding them gives us the density profile

$$n_{plasma} \simeq n_0 \exp\left(-\frac{m_i \varphi_g(\mathbf{r})}{2kT}\right), \qquad \text{where } \mathbf{g} = -\nabla\varphi_g \tag{2.25}$$

The ambipolar field therefore has the effect of dividing the mass of the ions by 2 and giving the electrons an effective mass $m_i/2$ in order to maintain electro-neutrality. The height scale of a plasma atmosphere is therefore doubled compared to that of a neutral gas.

2.3 Time response : plasma as a dielectric medium

2.3.1 Reminder on linear media

After studying plasma in the quasi-static regime, we now seek to study its response to a temporal disturbance – this will by the way help us clarify the domain of validity of the quasi-static regime. It is very convenient to study this time varying regime, for small disturbances, by introducing the dielectric response $\epsilon(\omega)$ of the plasma. This is the approach that we use in this course. We remind here a few useful relations that will be used in the following.

Polarisation vector, internal charge and current densities

We recall Maxwell-Gauss's equation in a medium,

$$\text{div } \mathbf{D} = \rho_{ext}, \qquad \text{where } \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{2.26}$$

and $\mathbf{P}$ is the volume density of dipoles in the medium. The internal charge density (polarisation charges) in the medium is (according to Maxwell-Gauss) related to $\mathbf{P}$ by

$$\rho_{int} = -\text{div } \mathbf{P} \tag{2.27}$$

The conservation of charge in the medium is written as

$$\frac{\partial \rho_{int}}{\partial t} + \text{div } \mathbf{j}_{int} = 0 \iff \text{div } \left(-\partial_t \mathbf{P} + \mathbf{j}_{int}\right) = 0. \tag{2.28}$$

The current density in the medium is therefore related to **P** by

$$\mathbf{j}_{int} = \frac{\partial \mathbf{P}}{\partial t}. \tag{2.29}$$

Linear media

The medium is linear if there exists a tensor $\boldsymbol{\chi}$, called *electrical susceptibility*, such that

$$P_i(t) = \varepsilon_0 \int \chi_{ij}(t-t')E_j(t')dt'. \tag{2.30}$$

This relationship is expressed in a much simpler way in Fourier space,

$$\mathbf{P}_\omega = \varepsilon_0 \boldsymbol{\chi}_\omega \cdot \mathbf{E}_\omega \tag{2.31}$$

In such a linear medium, the electric induction vector **D** is related to the electric field by

$$\mathbf{D}_\omega = \varepsilon_0 \left(\mathbf{I} + \boldsymbol{\chi}_\omega\right) \cdot \mathbf{E}_\omega \tag{2.32}$$

and we define the dielectric tensor of the medium by

$$\mathbf{D}_\omega = \boldsymbol{\varepsilon}_\omega \cdot \mathbf{E}_\omega \qquad \boldsymbol{\varepsilon}_\omega \equiv \epsilon_0 \left(\mathbf{I} + \boldsymbol{\chi}_\omega\right) \tag{2.33}$$

Response to an external perturbation

In the absence of an external field, or external charge density in the continuous medium, Maxwell-Gauss's equation reduces to (the right-hand side of the equations makes the electrostatic assumption, i.e. **rot E** $= 0$)

$$\text{div } \mathbf{D}_\omega = 0 \xrightarrow{\text{E.S}} \boldsymbol{\varepsilon}_\omega \cdot \mathbf{E}_\omega = 0 \tag{2.34}$$

In the presence of an oscillating external field $\mathbf{E}_{ext,\omega}$, this gives

$$\text{div } \mathbf{D}_\omega = \varepsilon_0 \text{div } \mathbf{E}_{ext,\omega} \xrightarrow{\text{E.S}} \boldsymbol{\varepsilon}_\omega \cdot \mathbf{E}_\omega = \varepsilon_0 \mathbf{E}_{ext,\omega} \tag{2.35}$$

These latter relations are useful for evaluating the response of a medium to a longitudinal electric field.

2.3.2 Inertial response : oscillation and plasma resonance

In the presence of a time-varying electric field, the plasma becomes polarised, since, because of their different masses, electrons and ions respond differently to this field. The dynamic response of species s to the electric field in the plasma $\tilde{\mathbf{E}}$ is given (neglecting any thermal effects for the moment) by

$$\frac{d^2\tilde{\mathbf{r}}_s}{dt^2} = \frac{q_s}{m_s}\tilde{\mathbf{E}}(t), \qquad \text{or } \tilde{\mathbf{r}}_{s,\omega} = -\frac{q_s}{m_s\omega^2}\tilde{\mathbf{E}}_\omega \tag{2.36}$$

Each charge creates a small oscillating dipole $\tilde{\mathbf{p}}_s = q_s\tilde{\mathbf{r}}_s$. The density of these dipoles is given by

$$\tilde{\mathbf{P}}_\omega = \sum_s n_s\tilde{\mathbf{p}}_s \simeq \sum_s n_{0,s}\tilde{\mathbf{p}}_s = -\sum_s \frac{n_{0,s}q_s^2}{m_s\omega^2}\tilde{\mathbf{E}}_\omega, \tag{2.37}$$

which we rewrite as

$$\tilde{\mathbf{P}}_\omega = -\varepsilon_0\sum_s \frac{\omega_{p,s}^2}{\omega^2}\tilde{\mathbf{E}}_\omega, \quad \text{where } \omega_{p,s}^2 = \frac{n_{0,s}q_s^2}{m_s\varepsilon_0} \tag{2.38}$$

where we have introduced the plasma frequency ω_p. We have therefore derived the plasma susceptibility,

$$\boldsymbol{\chi}_\omega = -\sum_s \frac{\omega_{p,s}^2}{\omega^2} \simeq -\frac{\omega_{p,e}^2}{\omega^2} \tag{2.39}$$

since the electron mass is very small compared to the ion mass.

In the absence of an external field, and under the electrostatic assumption, the field in the plasma satisfies the relation (2.34). The tensor $\boldsymbol{\varepsilon}_\omega$ is scalar here, which simplifies the relation ; we have

$$\varepsilon_\omega\mathbf{E}_\omega = 0. \tag{2.40}$$

This relation can be verified in two ways : trivially, the field can be zero, $\mathbf{E}_\omega = 0$ in the plasma. But this relation also informs us of a non-trivial phenomenon : a non-zero longitudinal field can exist in the plasma, provided that

$$\varepsilon_\omega = 0 \iff \omega^2 = \omega_p^2 \tag{2.41}$$

This phenomenon corresponds to what is called *plasma oscillation*. We can understand the nature of this oscillation by writing the space-time form of the field

$$(\omega^2 - \omega_p^2)\mathbf{E}_\omega = 0 \iff \left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right)\mathbf{E}(\mathbf{r},t) = 0 \iff \mathbf{E}(\mathbf{r},t) = Re\sum_{\mathbf{k}}\mathbf{E}_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_p t)} \tag{2.42}$$

where the spatial component of the field is expressed as a sum of modes $\mathbf{E}_0(\mathbf{r}) = \sum_k \mathbf{E}_k \exp(i\mathbf{k}\cdot\mathbf{r})$. This form shows this oscillation as a sum of plane waves that are decoupled from each other, with zero group velocity (since there is no relationship between ω and $\mathbf{k}$) ; this is why we do not refer to a plasma wave in this case, but rather a plasma oscillation. This is accompanied by a charge density oscillation,

$$\rho(\mathbf{r},t) = \epsilon_0 \mathrm{div}\ \mathbf{E}(\mathbf{r},t) = Re \sum_{\mathbf{k}} i\epsilon_0 \mathbf{k}\cdot\mathbf{E}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_p t)}, \tag{2.43}$$

which we can see is in phase quadrature with the electric field oscillation.

Finally, let us consider the response of the plasma to the application of an external electric field $E(t) = E_0 e^{-i\omega t}$. The total field in the plasma is related to the applied field by eq.(2.35). We have

$$\varepsilon_\omega \cdot \mathbf{E}_\omega = \varepsilon_0 \mathbf{E}_{ext,\omega}, \tag{2.44}$$

which tells us about the response of the plasma along the applied external field. For this, we have

$$E_\omega = \frac{\omega^2 E_0}{\omega^2 - \omega_p^2}. \tag{2.45}$$

This relationship illustrates several important phenomena :

- In the high frequency limit $\omega \gg \omega_p$, the field in the plasma is equal to the external field : the plasma behaves like a vacuum.
- In the low frequency limit $\omega \ll \omega_p$, the field in the plasma is zero : the plasma completely shields the external field
- The field in the plasma diverges at the plasma frequency : this acts as the resonance frequency of the medium. The divergence is actually controlled by effects not taken into account here (collisions or other kinetic effects).

2.3.3 Plasma oscillation : a heuristic view

Here we present a more visual representation of plasma oscillation. Consider a quasi-neutral plasma of electrons and ions, each with a density n_0 and a charge $\pm e$. We move a layer of electrons with width L by a small displacement $\Delta x \ll L$, thus creating two regions of non-zero charge density (see fig.2.3). The system is assumed to be invariant under translation in the two directions perpendicular to the x axis, so that all quantities depend only on the x coordinate. We seek the temporal evolution of this initial configuration and assume that we can consider the ions to be stationary (we will verify at the end that this is indeed the case).

The electric field in the plasma is given by Maxwell-Gauss's equation,

$$\frac{dE}{dx} = -\frac{e(n_0 - n_e(x))}{\varepsilon_0} \Rightarrow E(x) \sim en_0\Delta x/\epsilon_0 \tag{2.46}$$

in most of the region between 0 and L, since we assume that Δx is a very small quantity compared to L. Outside the region $0 < x < L$, the electric field is zero. Inside this region,

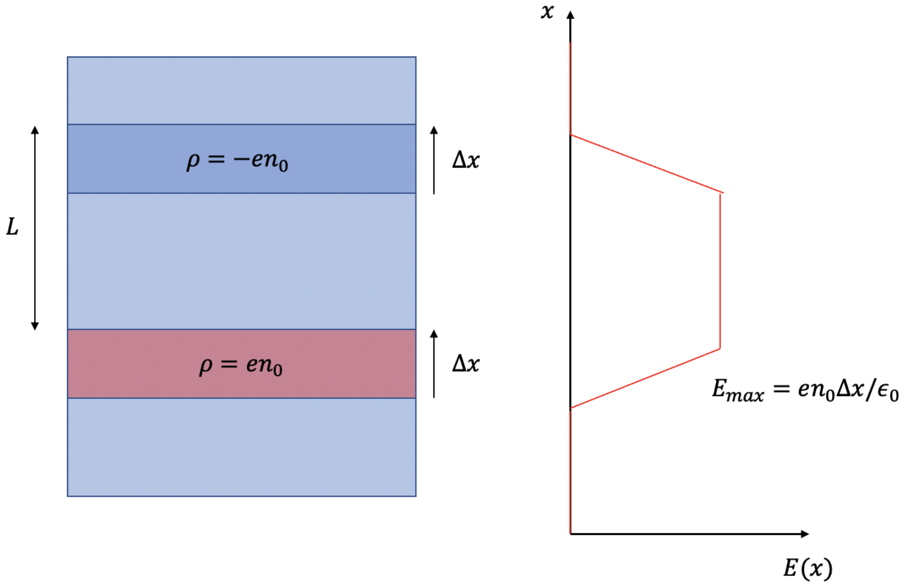

FIGURE 2.3 – Illustrated representation of plasma oscillation.

the dynamics of the electrons is given by

$$\frac{d^2\Delta x}{dt^2} = -\frac{eE}{m} = -\frac{e^2 n_0}{m\epsilon_0}\Delta x. \tag{2.47}$$

The electrons therefore oscillate at the plasma frequency $\omega_p = (n_0 e^2/m\epsilon_0)^{1/2}$. The electric field in the plasma is

$$E(0 < x < L, t) \simeq \frac{en_0\Delta x_0}{\epsilon_0}\cos\omega_p t \tag{2.48}$$

Under the action of this electric field, the amplitude of the ion oscillation is given by

$$\Delta x_{ions} \sim \frac{e^2 n_0}{m_i\epsilon_0\omega_p^2}\Delta x_0 = \frac{\omega_{p,ions}^2}{\omega_p^2}\Delta x_0 = \frac{m_e}{m_i}\Delta x_0 \tag{2.49}$$

which is smaller than the amplitude of the electron motion by a factor of at least ~ 2000. This justifies the assumption of immobile ions.

2.3.4 Transverse oscillation : the electromagnetic wave in a plasma

We discussed the case of electrostatic oscillation (div $\mathbf{E} = 0$, **rot** $\mathbf{E} \neq 0$) in the previous section. Let us now consider the case of transverse oscillation (div $\mathbf{E} = 0$,

rot $\mathbf{E} \neq 0$). Considering the space-time transform of Maxwell's equations, we have

$$i\mathbf{k} \times \mathbf{E}_{\omega\mathbf{k}} = i\omega \mathbf{B}_{\omega\mathbf{k}} \qquad \text{Maxwell-Faraday} \tag{2.50}$$

and

$$i\mathbf{k} \times \mathbf{B}_{\omega\mathbf{k}} = \mu_0(\mathbf{j}_{\omega\mathbf{k}} - i\omega\varepsilon_0\mathbf{E}_{\omega\mathbf{k}}) \qquad \text{Maxwell-Ampère} \tag{2.51}$$

The current is related to the plasma polarisation vector by the relation (2.29), so Maxwell-Ampère can be rewritten as

$$i\mathbf{k} \times \mathbf{B}_{\omega\mathbf{k}} = -i\omega\mu_0(\mathbf{P}_{\omega\mathbf{k}} + \varepsilon_0\mathbf{E}_{\omega\mathbf{k}}) = -i\frac{\omega}{c^2}\boldsymbol{\varepsilon}_{\boldsymbol{r},\boldsymbol{\omega}\mathbf{k}} \cdot \mathbf{E}_{\omega\mathbf{k}} \tag{2.52}$$

where we have used $\varepsilon_0\mu_0 = 1/c^2$ and introduced the relative permittivity $\boldsymbol{\epsilon_r} = \boldsymbol{\varepsilon_r}/\varepsilon_0$ Substituting back into Faraday's equation, we obtain

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_{\omega\mathbf{k}}) = -k^2\mathbf{E}_{\omega\mathbf{k}} = -\frac{\omega^2}{c^2}\boldsymbol{\varepsilon}_{\boldsymbol{r},\boldsymbol{\omega}\mathbf{k}} \cdot \mathbf{E}_{\omega\mathbf{k}} \tag{2.53}$$

where we have used $\mathbf{k} \cdot \mathbf{E} = 0$. This is the general equation for the propagation of a transverse wave in a dielectric medium. To obtain the properties of the wave in a plasma, we may simply use the "inertial" expression of $\varepsilon_{\omega\mathbf{k}}$ derived in paragraph 2.3.2,

$$\left(\omega^2 - \omega_p^2 - k^2c^2\right)\mathbf{E}_{\omega\mathbf{k}} = 0. \tag{2.54}$$

The dispersion relation of the electromagnetic wave in a plasma is thus

$$\omega^2 = \omega_p^2 + k^2c^2, \tag{2.55}$$

showing that transverse waves whose frequencies are lower than the plasma frequency are evanescent (k imaginary). It also shows that the phase velocity $v_\varphi = \omega/k$ of electromagnetic waves in a plasma is greater than the speed of light [3]. This produces a notable effect : when an electromagnetic ray enters a plasma interface, it is refracted outwards at the normal to the surface (contrary to the behaviour of usual optical systems, which have an optical index greater than 1). The optical refractive index of plasma is

$$n = \frac{kc}{\omega} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} < 1. \tag{2.56}$$

2.4 Exercises

2.4.1 Capacitance of a conductive sphere in plasma

Consider a sphere of radius a in a plasma consisting of ions with charge $+e$ and density n_i and electrons with density n_e. The problem has spherical symmetry with

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[3] Of course, we can verify that the group velocity v_g is always less than c. You can demonstrate as an exercise that $v_\varphi v_g = c^2$

respect to the centre of the sphere, so the densities depend only on the distance r from this centre.

We want to calculate the capacitance of the sphere. To do this, we assume a device that maintains it at a potential V relative to "infinity" - the far-field potential being that of the undisturbed plasma, which we assume to be equal to 0.

1. Calculate the potential $\varphi(r)$ around the sphere, assuming a quasi-static regime for ions and electrons. (Hint : you can introduce $\psi(r) = r\varphi(r)$ to simplify Poisson's equation in spherical coordinates, and solve for ψ).
2. Calculate the charge carried by the sphere, assuming the overall neutrality of the sphere + plasma system.
3. Deduce the capacitance of the conductive sphere. How does it differ from the capacitance of a sphere in a vacuum ?
4. Discuss the validity of the quasi-static approximation.

2.4.2 Pannekoek-Rossland electric field

Consider a self-gravitating plasma sphere (a star...), with mass M and radius R. Assume spherical symmetry and a stationary problem.

1. Relate the electric field $\mathbf{E}(\mathbf{r})$ at any point on the star to the acceleration due to gravity $\mathbf{g}(\mathbf{r})$ at that same point.
2. Deduce the relationship between the total charge Q of the star and its mass M.
3. Give the expression for the electric field (known as the *Pannekoek-Rossland* field) outside the star.

2.4.3 Ambipolar diffusion

We consider the diffusive motion of a plasma cloud in an infinite, homogeneous, neutral gas of density n_n. We consider that collisions between plasma particles and the neutral gas produce a friction force $\mathbf{f}_\alpha = \nu_\alpha \mathbf{u}_\alpha$. With $\nu_\alpha \equiv v_{th,\alpha}/\lambda$ the collision frequency between neutrals and species α ($\alpha = i, e$, for ions and electrons, respectively, and λ is a constant (the mean free path), which we assume to be identical for all species). We assume that friction dominates convection, so that the motion of population α is described by the equation

$$0 = -kT\nabla n_\alpha + n_\alpha q_\alpha \mathbf{E}(\mathbf{r}) - n_\alpha \nu_\alpha m_\alpha \mathbf{u}_\alpha \tag{2.57}$$

1. Express the average velocity $\mathbf{u}_\alpha$ of each population.
2. We assume that the electric field is zero. In this case, express the diffusion coefficient of each population D_α, explicitly showing the mass m_α.

3. What problem do you identify ? What condition(s) must be imposed in order to solve this problem ?
4. What is the actual diffusion coefficient (known as *ambipolar*) of the plasma in the neutral gas ?

2.4.4 Dielectric of a hot plasma : oscillations and screening

In section 2.3.2, we calculated the value of the dielectric function ε_ω of the plasma in a very simplified case, neglecting any thermal effects in particular. Here, we seek to calculate the dielectric in the case where the electrons have a non-zero pressure, but retaining the simplifying assumption of immobile ions and a purely electrostatic problem. The electron population is characterised by its density n_e, its average velocity $\mathbf{u}_e$ and its pressure p_e. This latter is related to the density by a polytropic closure relation $d_t(p_e n_e^{-\gamma}) = 0$. The ion population is characterised by its density $n_i \equiv n_0$ and its charge $q_i = +e$.

1. Express the equations of the electron fluid (conservation of particle number and conservation of momentum).
2. Linearise these equations, as well as the closure equation, assuming that $n_e = n_0 + \tilde{n}_e(\mathbf{r}, t)$, $\mathbf{u}_e = \tilde{\mathbf{u}}_e(\mathbf{r}, t)$, $p_e = p_{e0} + \tilde{p}_e(r, t)$, $n_i = n_0$ and $\mathbf{E} = \tilde{\mathbf{E}}(\mathbf{r}, t)$. The quantities marked with tildes are "small perturbations".
3. Calculate the longitudinal susceptibility of the plasma in this case, $\chi^L_{\omega\mathbf{k}}$ (we will "algebraise" the equations by placing ourselves in Fourier space $(t, \mathbf{r} \to \omega, \mathbf{k})$, calculate the current $\mathbf{j}$, and assume that the field is longitudinal, i.e. $\mathbf{E} \parallel \mathbf{k}$).
4. How is the dispersion relation of plasma oscillations modified ? Why can we now talk about plasma waves ?
5. How is the dispersion relation of electromagnetic (transverse) waves in plasma modified ?
6. Comment on the low-frequency limit $\omega \to 0$ of the dielectric tensor : how is it related to the concept of electrostatic shielding ?

2.4.5 Low-frequency regime : the ionic acoustic wave

Consider an electrical disturbance with wave vector k and frequency ω. Since the thermal velocity of ions is generally very small compared to that of electrons, there is a frequency range such that

$$k v_{th,i} \ll \omega \ll k v_{th,e}. \tag{2.58}$$

We will study this regime in this exercise.

1. How can we characterise the response of ions and electrons in such a regime ?

2. Using the results from the previous exercise, express the dielectric tensor considering the motion of both populations (ions and electrons — unlike in the previous exercise where the ions were considered immobile). We will limit ourselves to the case of longitudinal fluctuations.
3. Simplify the dispersion relation obtained in the frequency regime under consideration.
4. We now assume $k\lambda_D \ll 1$: why can we talk about ionic acoustic waves ? What is the propagation velocity of these waves ? Comment on this in relation to the case of acoustic waves in a neutral gas.

2.4.6 Dielectric of a collisional plasma

Consider a plasma in which the electron population is "collisional", characterised by a collision frequency ν – the dynamics of the electrons are then given by

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{e}{m_e}\mathbf{E} - \nu\frac{d\mathbf{r}}{dt} \tag{2.59}$$

1. Calculate the dielectric function ϵ_ω of the plasma. How does the plasma respond to an imposed external field $E_0 sin(\omega t)$?
2. How is the dispersion relation of plasma oscillations modified ?
3. How is the dispersion relation of electromagnetic (transverse) waves in plasma modified ?

Chapter 3: Coulomb collisions

In the previous chapter, we studied collective phenomena, through which the field and a large number of particles organise themselves in a coherent manner ($\Gamma \ll 1$ implies correlation scales that are large compared to the interparticle distance). In this chapter, we study the "opposite" aspect of interactions between pairs of particles, i.e. collisions.

In fully ionised plasmas, these collisions are controlled by the long-range Coulomb force (as opposed to the short-range dipole-dipole interactions that control collisions between atoms or molecules). These collisions play an important role in understanding transport phenomena (electrical and thermal conduction, diffusion, etc.) in plasmas. We will also see that their relative inefficiency helps us understand why collective phenomena generally prevail in plasmas, justifying a non-collisional treatment of a large number of phenomena.

3.1 Cross sections

3.1.1 Large-angle collisions

The angle of deflection (in the center-of-mass frame) during a collision between two charged particles, with indices 1 and 2, is given by the *Rutherford formula* [1]

$$\tan\frac{\theta}{2} = \frac{q_1 q_2}{b\mu v^2} \tag{3.1}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass, v is the relative velocity between the two particles, and b is the impact parameter. In this course, in order to avoid overly complicated calculations, we will limit ourselves to the situation (of interest !) of the collision of a light particle (an electron) with infinitely massive particles (ions, with charge $+Ze$). In this case, the center-of-mass reference frame is simply the reference frame in which the ion is at rest, and the reduced mass is the mass of the electron $\mu = m_e$. Furthermore, there is no energy transfer between the two particles in this case : the modulus of the electron's velocity vector is constant, there is only a deflection of the trajectory.

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[1] In this chapter, we will use the notation $q_e \equiv e/\sqrt{4\pi\varepsilon_0}$

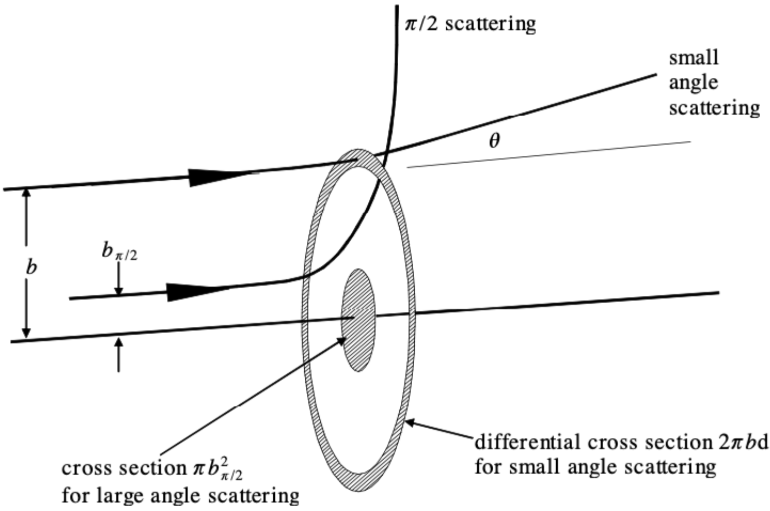

FIGURE 3.1 – Rutherford scattering

There is a deflection of 90° when the right-hand side is equal to 1, i.e. when the impact parameter is

$$b_{90}(v) = \frac{Zq_e^2}{m_e v^2} \tag{3.2}$$

This impact parameter is similar to the Landau length, which we introduced in the previous chapter, $\lambda_L = 4\pi q_e^2/kT$: we see that the Landau length gives (in order of magnitude) the impact parameter for which a collision between thermal particles will produce a large-angle deflection. We have seen that $\lambda_L \ll \ell$ (ℓ being the inter-particle distance) : we can therefore expect there to be few large-angle collisions in plasmas; if, for example, we calculate the angle of deviation of a thermal particle for an impact parameter equal to the typical inter-particle distance, we obtain $\theta \sim \lambda_L/\ell \sim \Gamma^{2/3}$. In the solar wind, $\Gamma \sim 10^{-10}$, so typically $\theta \sim 10^{-6}$ rad.

The large-angle collision cross section is defined as $\sigma_{l.a.} = \pi b_{90}^2$. Consider a test electron moving at velocity v in a "field of scattering particles" (ions at rest) with density n. The mean free path of such a particle with respect to large-angle collisions is $\lambda_{l.a.} = (\sigma_{l.a.} n)^{-1}$, and the associated collision frequency is

$$\nu_{l.a.} = v/\lambda = nv\pi b_{90}(v)^2 = \pi n \left(\frac{Zq_e^2}{m_e}\right)^2 \frac{1}{v^3} \tag{3.3}$$

Formula (3.3) gives the typical time over which the electron will undergo a large-angle collision (i.e. over which it will pass within a distance of less than b_{90} from an ion). It can be seen that this time scale decreases very sharply with particle velocity ; this is a general property of Coulomb collisions, which has important effects in plasma physics : the hotter a plasma is, the less collisional it is.

To conclude this paragraph, we can estimate the order of magnitude of the mean free path of a thermal particle for large-angle collisions,

$$\lambda_{l.a.} \sim (n\sigma_{l.a.})^{-1} \sim \frac{\ell^3}{\lambda_L^2} \sim \Gamma^{-2}\lambda_L \sim \Gamma^{-1}\lambda_D, \tag{3.4}$$

the Debye sphere is therefore not very collisional ! (at large angles at least) [2]

3.1.2 Small-angle collisions and Coulomb logarithm

We have estimated the cross section for large-angle collisions. However, in addition to the binary interactions producing these large angle deflections, an electron is also continuously deflected as a result of interactions with particles located at distances greater than b_{90} ; although these collisions generate small-angle deflections, they are much more numerous than large-angle deflections. We must therefore ask ourselves what mechanism dominates the total deflection of the particle : the rare large-angle collisions, or the sum of the very frequent small-angle collisions ?

From formula (3.1), the angle of deflection after a collision of large impact parameter is [3]

$$\theta \simeq -\frac{2b_{90}(v)}{b} \tag{3.5}$$

which corresponds to a change in the particle's velocity in the direction of $\mathbf{b}$ ($\mathbf{b}$ here being the impact parameter vector, and $\mathbf{u}_b$ the unit vector along $\mathbf{b}$)

$$\delta \mathbf{v}_\perp \simeq -v\theta \mathbf{u}_b = -\frac{2b_{90}(v)v}{b^2}\mathbf{b} \tag{3.6}$$

this last result being correct to first order in θ (otherwise a correction $\propto \theta^2$ must be included in the direction parallel to $\mathbf{v}$).

Let us consider the interaction of an electron moving at velocity v with respect to a background of infinitely massive ions at rest, of number density n.

We use cylindrical geometry, with the axis being the velocity vector $\mathbf{v}$ of the electron. The number of ions interacting with the electron during a time Δt, with a vector impact

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[2] Note that the orders of magnitude obtained are correct for collisions between electrons or between ions. The reduced mass is then $\mu = m_s/2$ ($s = e, i$), and a change of reference frame from the center of mass to the laboratory frame is required to obtain the exact angle of deviation, but these calculations give correction factors "of the order of unity".

[3] the "-" sign comes from the fact that in formula 3.1, we have $q_1 q_2 = -Zq_e^2$, the electron and ions having opposite charges. The deflection therefore occurs "downwards" in Fig. 3.1

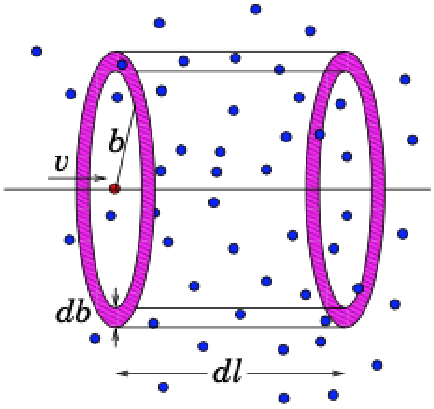

FIGURE 3.2 – Interaction of the test particle with the ion scattering centers

parameter whose modulus is between b and $b + db$ and whose orientation is between ϕ and $\phi + d\phi$ is (see illustration in Figure 3.2)

$$dN = d^2\mathbf{b}v\Delta t = v\Delta t.bdb.d\phi \tag{3.7}$$

The average deflection of the test particle over the time interval Δt is calculated by summing the deflections caused by all the particles in the plasma. We therefore integrate over all impact parameters :

$$\langle \Delta \mathbf{v}_\perp \rangle = \int \delta \mathbf{v}_\perp dN = 2nb_{90}(v)v^2 \int_0^{2\pi} \mathbf{u}_b(\phi)d\phi \int db = 0 \tag{3.8}$$

where $\mathbf{u}_b(\phi) = \mathbf{b}/b$ is the radial unit vector in polar (cylindrical) coordinates. The integral of this vector over ϕ is 0 : we see that the deflections of the velocity vector cancel each other out on average, which was to be expected given the symmetry of the problem.

But if $\Delta \mathbf{v}_\perp$ is zero on average, the variance $\langle \Delta \mathbf{v}_\perp^2 \rangle$ as we shall demonstrate it, increases over time. The value of the variance is given by (using the same procedure as above)

$$\langle \Delta v_\perp^2 \rangle = 8\pi n \Delta t b_{90}^2(v)v^3 \int \frac{db}{b}. \tag{3.9}$$

We note (with dismay...) that the integral (3.9), taken between 0 and ∞, diverges at both 0 and ∞. The divergence for $b \to \infty$ stems from the fact that the Coulomb interaction is long-range ($1/r^2$). But in a plasma, Debye screening shortens the effective range of the Coulomb force. Thus, the integral can be regularised "on physical grounds" by setting the upper bound of the integration equal to λ_D – assuming an effective interaction potential equal to zero after this distance. The integral also diverges for $b \to 0$. This is explained by the fact that the angles of deviation become infinite for a zero value of the impact parameter. This (incorrect) result stems from the fact that we used an approximation of $\delta \mathbf{v}_\perp$ that is only valid for small angles. We therefore cut the integral for small values of b at $b_{90}(v)$. We finally obtain for the variation,

$$\frac{\langle \Delta v_\perp^2 \rangle}{\Delta t} = 8\pi n b_{90}^2(v) v^3 \ln \Lambda \tag{3.10}$$

We have introduced Coulomb's logarithm

$$\ln \Lambda = \ln \lambda_D / b_{90} \sim \ln \Gamma^{-1} \tag{3.11}$$

In most plasmas (natural or artificial), the thermal value of Coulomb's logarithm is $\ln \Lambda \sim 15 - 25$ for thermal particles.

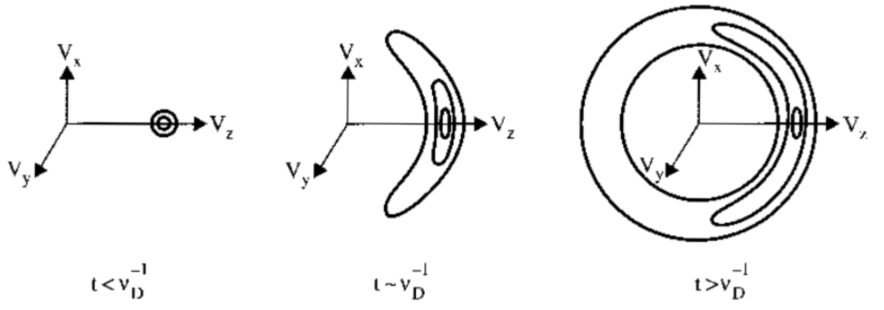

FIGURE 3.3 – Angular diffusion on a characteristic time scale ν_D^{-1} (Trubnikov 1965).

We have just shown that the variance of the deviation angle $\theta = v_\perp / v$ increases linearly with time. This is characteristic of an angular diffusion process : the particle undergoes many small random deviations. The effect is not a change in the average direction of the test particle's velocity vector, but an increase in the probability of finding this vector in a direction increasingly distant from its initial direction (see Figure 3.3). The diffusion coefficient characterising this process is

$$D_\theta \equiv \frac{\langle \Delta \theta^2 \rangle}{2\Delta t} = 4\pi n b_{90}^2(v) v \ln \Lambda = \nu_{ei} \left(\frac{v_{th}}{v} \right)^3 \tag{3.12}$$

where we have introduced the thermal electron-ion collision frequency

$$\nu_{ei} = \frac{4\pi n Z^2 q_e^4 \ln \Lambda}{m_e^2 v_{th}^3} \tag{3.13}$$

The collision frequency is thus defined, for small-angle Coulomb collisions, as the inverse of the time required for the scattering effect to substantially broaden the angular distribution of particle velocities.

Finally, note that the collision frequency obtained ν_{ei} is equal to that obtained for large-angle collisions in the first section, multiplied by Coulomb's logarithm $\ln\Lambda$ (and a factor of 4...). This gives another interpretation of the Coulomb logarithm (and the plasma parameter Γ) as the ratio between the efficiency of small-angle collisions and that of large-angle collisions. Since $\ln\Lambda \sim 20$, we have demonstrated that small-angle scattering is completely dominant in a fully ionised plasma.

3.2 Dynamical friction force and related effects

The angular scattering that has just been described is necessarily accompanied by a slowing down (a friction) of the electron in the parallel direction. This comes from the fact that the total energy $\mathscr{E}$ of the electron is conserved in the scattering process. One has

$$\Delta\mathscr{E} = \frac{1}{2}m_e\left((v+\Delta v_{\|})^2 + \Delta v_{\perp}^2 - v^2\right) = 0 \tag{3.14}$$

from which we have,

$$2v\Delta v_{\|} + \Delta v_{\|}^2 + \Delta v_{\perp}^2 = 0. \tag{3.15}$$

Since $\Delta v_{\|} \ll v$, the first term is much larger than the second one, which can be neglected. Taking the average of the remaining terms, we get the expression of the parallel slowing down

$$\frac{\langle\Delta v_{\|}\rangle}{\Delta t} = -\frac{1}{v}\frac{\langle\Delta v_{\perp}^2\rangle}{2\Delta t} = -\nu_{ei}\frac{v_{th}^3}{v^2} \tag{3.16}$$

Since this parallel component is applied along the electron velocity vector $\mathbf{v}$, the friction force on the electron due to the angular scattering can be expressed as

$$\mathbf{f} = \frac{m_e\langle\Delta v_{\|}\rangle}{\Delta t}\frac{\mathbf{v}}{v} = -m_e\nu_{ei}\frac{v_{th}^3}{v^3}\mathbf{v} \tag{3.17}$$

This is a general result : the existence of angular scattering with a diffusion coefficient D_α produces a dynamical friction force on the scattered particles $\mathbf{f} = -m_e D_\alpha \mathbf{v}$. We now investigate a few interesting effects directly linked to the existence of this friction force.

3.2.1 Runaway electrons and the Dreicer electric field

An important collisional process in plasma physics is called the runaway effect. It is due to the fact that the collision frequency decreases strongly with the velocity of the electron ($\propto v^{-3}$). Therefore, an electron in an electric field may gain in average energy from the field in spite of the collisions. The proper description of this effect necessitate a kinetic treatment ; we propose here a simplified depiction, providing correct orders of magnitude.

Consider an electron in a constant electric field $\mathbf{E}$, colliding with background ions. Its kinetic energy (actually its mean "directed" kinetic energy), checks

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -e\mathbf{E}\cdot\mathbf{v} - m_e\nu_{ei}\frac{v_{th}^3}{v}. \tag{3.18}$$

from which we see that an electron having a velocity v_{lim} such that

$$v_{lim}(E) = \left(\frac{m_e \nu_{ei} v_{th}^3}{eE} \right)^{1/2} \tag{3.19}$$

will neither lose nor gain energy from the field (it is somehow in equilibrium). An electron with a velocity $v < v_{lim}$ will be "overdamped", which means that its velocity will in average decrease under the action of the collisional friction. A particle with a velocity $v > v_{lim}$ will be "underdamped", which means that its energy will increase in time without limit under the action of the electric field. Such electrons are called "runaway" electrons. Note that, as small as the applied electric field can be, there will always be a small fraction of runaway electrons in the tail of the velocity distribution function.

The Dreicer electric field is the value of the electric field for which the thermal electrons become themselves runaway – at this stage no stable plasma can really exist. Its value is given by $v_{lim}(E_D) = v_{th}$,

$$E_D = \frac{m_e \nu_{ei} v_{th}}{e}. \tag{3.20}$$

We can now conveniently express $v_{lim}(E) = (E_D/E)^{1/2} v_{th}$.

Runaway electrons are important in plasma physics, since they can take a lot of energy out of the electric field (they can be responsible for the breaking of plasma state in lab experiments). They also appear to play an important (although not clearly elucidated) role in lots of astrophysical plasmas.

3.2.2 Subsonic fluid friction and electrical conductivity

We want to calculate the friction force $\mathbf{f}$ acting, not on a single particle, but on a small electron fluid volume of density n_e and mean velocity $\mathbf{u}$. This is

$$\mathbf{f} = -m_e n_e \langle \nu_{ei} \frac{v_{th}^3}{v^3} \mathbf{v} \rangle = -m_e n_e \nu_{ei} v_{th}^3 \int \frac{\mathbf{v} d^3\mathbf{v}}{v^3} f(\mathbf{v}) \tag{3.21}$$

where $f(\mathbf{v})$ is the distribution of the electron fluid velocities, so that the probability to find an electron with a speed between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ is $dp = f(\mathbf{v}) d^3\mathbf{v}$. We assume that our fluid element is near-equilibrium, so that $f(\mathbf{v})$ is a Gaussian with thermal speed v_{th} and a drift speed $\mathbf{u}$ such that $u \ll v_{th}$. We have

$$f(\mathbf{v}) = \frac{1}{(2\pi)^{3/2} v_{th}^3} e^{-(\mathbf{v}-\mathbf{u})^2/2v_{th}^2} \simeq \frac{1}{(2\pi)^{3/2} v_{th}^3} e^{-v^2/2v_{th}^2} \left(1 + \frac{\mathbf{v} \cdot \mathbf{u}}{v_{th}^2} \right) \tag{3.22}$$

The integral on the first term is equal to 0, since the integrand is odd. Let's define z along the vector $\mathbf{u}$. The x and y components of $\mathbf{f}$ are equal to 0 (odd integrands). So the only component left is along z, and is equal to

$$f_z = -\frac{m_e n_e \nu_{ei} u}{(2\pi)^{3/2}} \int \frac{v_z^2 d^3v}{v_{th}^2 v^3} e^{-v^2/2v_{th}^2} \tag{3.23}$$

The integral is calculated in spherical coordinates (with the convenient change of variable $\mu = \cos\theta$, so $d^3v = v^2 dv d\mu d\phi$),

$$\int_0^{2\pi} d\phi \int_{-1}^{1} \mu^2 d\mu \int_0^{\infty} \frac{vdv}{v_{th}^2} e^{-v^2/2v_{th}^2} = 2\pi \times \frac{2}{3} \times 1 = \frac{4\pi}{3}. \tag{3.24}$$

We finally obtain

$$\mathbf{f} = -m_e n_e \frac{2\nu_{ei}}{3\sqrt{2\pi}} \mathbf{u} \equiv -\frac{m_e n_e \mathbf{u}}{\tau_{ei}}. \tag{3.25}$$

So, there is a factor $2/3\sqrt{2\pi}$ between the viscous force on the fluid and the force on a single electron that would have a velocity equal to the thermal velocity. In several references[4], the friction timescale $\tau_{ei} = 3\sqrt{2\pi}/2\nu_{ei}$ is introduced, as in the previous equation.

The expression of this viscous force makes it possible to calculate the electric conductivity σ of a plasma, at least in the small electric field limit. We consider the ions at rest, so that the current density and the electric field are linked by $\mathbf{j} = -en\mathbf{u} = \sigma\mathbf{E}$. We consider values of the electric field much smaller than E_D, so that we can neglect runaway effects : all the conduction electrons are assumed to be underdamped, so that a steady state can be reached in the plasma. The equation describing this steady state is

$$0 = -e\mathbf{E} - m_e\mathbf{u}/\tau_{ei}, \tag{3.26}$$

so that the steady-state value of the fluid velocity is

$$\mathbf{u} = -\frac{e\tau_{ei}}{m_e}\mathbf{E} \tag{3.27}$$

from which we find the value of the plasma conductivity

$$\sigma = \frac{ne^2\tau_{ei}}{m_e} \tag{3.28}$$

Space plasmas are, to rare exceptions, never really close to equilibrium (the collisional mean-free paths are usually, for a large part of the electron energy distributions, much larger than the typical gradient scales of the system considered). The expressions of the conductivity given here is thus to be taken very cautiously.

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[4] e.g. the reference review *Transport processes in plasmas*, S.I. Braginskii, 1965

3.3 Examples and exercises

3.3.1 Ionization cross section*

Although a bit outside of the Coulomb collision topic, it is interesting to note that we can use a procedure similar to the one previously employed in order to derive the Thomson formula, seen in the first chapter, for the classical ionization cross section of an atom by electron impact.

We consider the energy transfer from an electron moving in a straight line with impact parameter b and energy $E = m_e v^2/2$ with respect to an electron bound to an atom situated at the origin of the coordinate system. Equation (3.6) gives us $\delta v_\perp$, from which we can calculate $\delta\varepsilon = m_e \delta v_\perp^2/2$. We have

$$\delta\varepsilon(E, b) = \frac{q_e^4}{Eb^2} \tag{3.29}$$

We can express the differential cross section

$$d\sigma = 2\pi b db = \frac{\pi q_e^4}{E\delta\varepsilon^2} d\delta\varepsilon. \tag{3.30}$$

For ionisation to occur, we need the energy transferred $\delta\varepsilon$ to be larger than the first ionisation energy W. On the other hand, the energy transferred cannot be larger than the initial energy E of the electron. The ionisation cross section is obtained by integrating the differential cross-section between these to energies,

$$\sigma_I = \frac{\pi q_e^4}{E} \int_W^E \frac{d\delta\varepsilon}{\delta\varepsilon^2} = \pi q_e^4 \frac{E - W}{E^2 W} = \frac{\pi q_e^4}{W^2} \left(\frac{W}{E} - \frac{W^2}{E^2} \right) \tag{3.31}$$

which is the Thomson formula and is illustrated on Fig.3.4.

3.3.2 Exercise : Slowing down of a fast ion by cold electrons

The problem of angular scattering of the electrons treated in the previous section is complementary to the problem a fast ion travelling in a cold electron population. The ion, through its motion, will induce small changes in the electron velocities, and will therefore transfer some of its energy to the electron population. We assume that the ion moves at a constant velocity $\mathbf{v}_{ion}$.

1. Calculate the energy transfer transferred from the ion to an electron, during the interaction with a single electron.
2. Calculate the energy transfer per unit time, integrating on all electrons.
3. What is the slowing down timescale ν_{ie} ? Compare to the electron scattering frequency ν_{ei}.

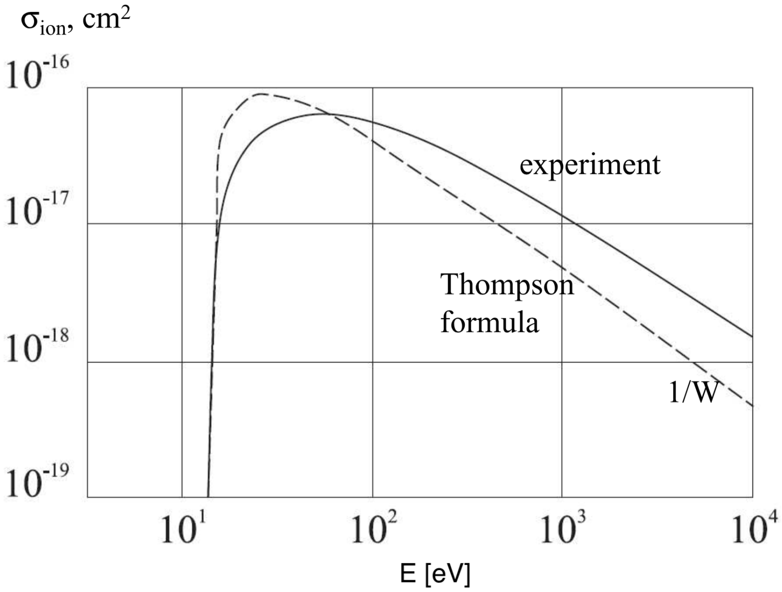

FIGURE 3.4 – Ionzation cross section as a function of the incident electron's energy and comparison of the classical formula to experiment.

3.3.3 Exercise : ambipolar electric field and Dreicer limit

Consider a gravitationally confined plasma. We have seen that an ambipolar electric field must exist to ensure quasi-neutrality in such a plasma. Under what conditions would this electric field be larger than the Dreicer field ? What would happen then ?

Chapter 4: Charging of a macroscopic conductive object in a plasma*

This chapter will not seen during the course and is not at the program of the exam

A plasma is composed of charged particles, animated by a thermal motion. An macroscopic object immersed in a plasma will, as a consequence, collect thermal electric currents onto its surfaces. So, the object will charge itself, and reach some electric potential, that will in turn modify the values of the currents.

The total charge Q of the object is determined by the equation

$$\frac{dQ}{dt} = I_e + I_i + I_{ph} + I_{sec} + ... \tag{4.1}$$

where the right-hand side sums the different current resulting from different processes (from left to right : electron current, ion current, photo-electron emission current, secondary electron emission current, others...). Note that the orientation of the currents is chosen positive for currents ongoing to the object. The equilibrium charge will be determined by the condition $dQ/dt = 0$; so, it is reached when all the currents on the object cancel each other.

4.1 Expression for the currents

We develop here a simplified but practical model for the electric currents. We assume that the object's surface is at a position $z = 0$ and is infinite along the directions x and y of a cartesian frame. We assume that the object has a potential φ, and introduce the subscript $\alpha = i, e$ to refer to a plasma population.

4.1.1 Plasma currents

If $q_\alpha \varphi < 0$, then the potential is attractive and all the particles can reach the surface. Assuming that the velocity distribution of the specie α is a Maxwellian, the current onto the surface is given by

$$I_\alpha = q_\alpha S \int_0^\infty \frac{n_\alpha}{\sqrt{2\pi} v_{th,\alpha}} e^{-v_z^2/2v_{th,\alpha}^2} v_z dv_z = I_{\alpha,0} \tag{4.2}$$

where

$$I_{\alpha,0} = q_\alpha n_\alpha v_\alpha S \tag{4.3}$$

here we introduced $v_\alpha = (kT_\alpha/2\pi m_\alpha)^{1/2}$, S is the surface of the object and the thermal velocity is $v_{th,\alpha} = \sqrt{kT_\alpha/m_\alpha}$).

Now if $q_\alpha\varphi > 0$, the potential is repulsive and only the particles having a z component of their velocity vector larger than $\sqrt{2q_\alpha\varphi/m_\alpha}$ can reach the surface of the object. The expression of the current onto the surface S is

$$I_\alpha(\varphi) = q_\alpha S \int_{\sqrt{2q_\alpha\varphi/m_\alpha}}^{\infty} \frac{n_\alpha}{\sqrt{2\pi}v_{th,\alpha}} e^{-v_z^2/2v_{th,\alpha}^2} v_z dv_z = I_{\alpha,0} \exp\left(-\frac{q_\alpha\varphi}{kT_\alpha}\right) \tag{4.4}$$

Note that to obtain these expressions, we neglected a possible drift velocity of the charged population with respect to the surface, and consider only the thermal motion. This is nearly always a very good approximation for electrons, but in general not true for ions, for two reasons :

— In space environments, ions flow will in general be supersonic (the solar wind ion fluid is supersonic, and orbital velocities of spacecraft for instance in the ionosphere are in general quite larger than the ion thermal speeds). The ion speed to consider in the expression of the current is then $v_i = u_i$, the drift speed of ions with respect to the surface.

— It can be shown that the ions (if the plasma flow to the surface is initially subsonic, typically what happens in laboratory electrostatic discharges) must enter the sheath at the Bohm speed[1] $v_i = v_B = \sqrt{kT_e/m_i}$ in order for a stable, steady-state sheath to be maintained.

4.1.2 Photoelectron and secondary electron currents

Some charging currents do not originate from the plasma, but from the charged surface itself. It is the case when the surface is illuminated by ionizing radiation, and emits photo-electrons or secondary electrons. Here, we discuss the case of photoelectrons and use the subscript "ph", but the discussion and expressions are exactly the same for emissions of secondary electrons.

In the case of a repulsive potential $\varphi < 0$, all of the electrons will leave the surface and the current will be

$$I_{ph} = eS_{lit}J_{ph,0} \tag{4.5}$$

where $J_{ph,0}$ is the photo electron flux at zero potential, which depends only on the surface illumination and on the physical properties of the illuminated material. S_{lit} is the surface receiving the UV light. Mind the sign of the expression, which comes from our

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[1] This is called the Bohm sheath criterion

convention of orientation of the currents, cf. eq.(4.1).

If $\varphi > 0$, the potential is attractive and only the fastest electrons will leave the surface. Others will be attracted back and not contribute to the escaping current. We have

$$I_{ph}(\varphi) = eS_{lit}J_{ph,0}\int_{\sqrt{2e\varphi/m_e}}^{\infty}\frac{1}{\sqrt{2\pi}v_{th,ph}}e^{-v_z^2/2v_{th,ph}^2}v_z dv_z = eS_{lit}J_{ph,0}\exp\left(-\frac{e\varphi}{kT_{ph}}\right). \tag{4.6}$$

The photoelectron distribution has been assumed to be Maxwellian with a temperature T_{ph}. Note that in reality, emitted photo or secondary electrons are usually not well modeled by a single Maxwellian, and multi-temperature models are often used.

4.2 Expressions of the floating potential

4.2.1 In a plasma : electron and ion current only

We consider the case were only the two first terms in the right-hand side of eq.(4.1) are of importance. We can see that, because of the small electron mass, we shall in general have $I_{e,0} \gg I_{i,0}$, and the object will tend to charge negatively. Therefore all protons will be collected (the ion current is then independent of the value of the potential, and is for this reason usually called the ion "saturation current"), whereas the electron current will depend on φ.

The potential reached in equilibrium is obtained from the condition $dQ/dt = 0$,

$$\varphi_{eq} = \frac{kT_e}{e}\ln\left(\frac{v_i}{v_e}\right) = -\frac{kT_e}{2e}\ln\left(\frac{m_i}{2\pi m_e}\right) \tag{4.7}$$

where the last equality assumes an ion current given by the Bohm criterion (typical case for laboratory measurements). In the case of a supersonic flow, v_i must be replaced by u_i and the ion mass does not appear in the expression any more.

The object then carries an equilibrium charge $Q_{eq} = C\varphi_{eq}$, with C the capacitance of the object. The equilibrium potential is then a few times the plasma electron temperature.

4.2.2 A sunlit surface : objects in the interplanetary space

In the vicinity of the Sun, objects receive ionizing solar UV. For an order of magnitude at 1 AU, $J_{ph,0} \sim 50$ μA/m^2, which is much larger than the typical electron current from the interplanetary "solar wind" plasma onto the object (which is $I_{e,0}/S \sim 0.5$ μA/m^2 at

1 AU). Therefore in typical interplanetary conditions, an object is charged positively.

We can obtain the equilibrium potential of, say, a spacecraft in the interplanetary space by using eq.(4.1), and neglecting the ion current

$$\varphi_{eq} = \frac{kT_{ph}}{e} \ln \left(\frac{J_{ph,0} S_{lit}}{e n_e v_e S} \right) \tag{4.8}$$

which is a few times the photo-electron temperature expressed in eV. For typical solar wind conditions, $\varphi_{eq} \sim 5 - 10$ V. Interestingly this value does not depend much on the distance from the Sun, since J_{ph_0} and n_e both vary as the inverse square of the distance from the Sun, so their ratio is approximately a constant.

4.2.3 Charge of a dust grain in the interplanetary medium

This has interesting consequences for the physics of dust grains in the interplanetary medium. The charge of a dust of size a is $q \sim C\varphi_{eq}$, with φ_{eq} given by eq.(4.8) and $C \simeq 4\pi\varepsilon_0 a$ being the capacitance of a sphere of radius a – which is a good approximation since a is much smaller than the local Debye length (which is of the order of 10 m in the solar wind at 1 AU).

Therefore the charge carried by a dust grain varies linearly with is size, $Q \sim a$. On the other hand, the mass of a dust grain is proportional to its volume, so the charge on mass ratio of a grain is inversely proportional to the square of its size, $Q/m \sim a^{-2}$.

This has important consequences on the interplanetary dust cloud dynamics : small dust grain have an important charge on mass ratio, and their dynamics will be strongly influenced by the Lorentz force (they will behave as very heavy ions) whereas large grains will be influenced by gravitational force only and have roughly Keplerian orbits.

4.3 Principle of the Langmuir probe

We can now understand the working principle of an important plasma sounding device : the Langmuir probe. The idea is to place a conducting device in a plasma and to bias it at some potential Φ_B. Measuring the intensity $I(\Phi_B)$ flowing through the device will let us estimate the density and temperature of the plasma.

In a plasma, without photoelectron emission effect, the characteristic curve looks like the one presented in Fig.4.1, and can be interpreted as follows (the orientation of the current is toward the plasma) :

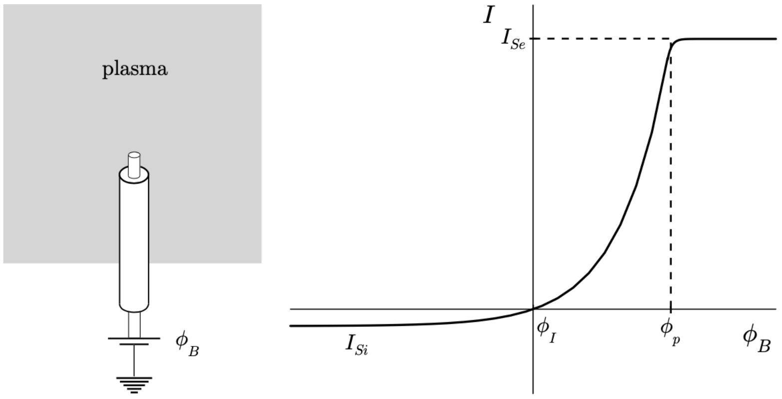

FIGURE 4.1 – Typical characteristic curve of a Langmuir probe, showing the current I drawn from the probe as a function of the applied bias voltage Φ_B. Labels are Φ_f, floating potential ; Φ_p, plasma potential – in our model, $\Phi_p = 0$; I_{Si}, ion saturation current ; I_{Se}, electron saturation current. From *J.D. Callen, Fundamentals of plasma physics, 2003.* Note that the convention for the orientation of the current is upward (to the plasma, opposite from the convention of the first section of this chapter).

— At $\Phi_B \to -\infty$, the collected current is almost completely ionic and is the ion saturation current $I_{Si} = -n_0 e v_i$, where v_i is the ion velocity at the entrance of the sheath. In a steady laboratory plasma experiment (which the fig.4.1 illustrates), it is given by $v_i^2 = kT_e/m_i$.

— At $\Phi_B \to +\infty$, the collected current is almost completely electronic. It is the electron saturation current $I_{Se} = n_0 e v_e$, where v_e is typically the thermal agitation speed, $v_e^2 = kT_e/(2\pi m_e)$.

— At $I = 0$, the potential is by definition the floating potential, that the probe would have if let passively in the plasma, $\Phi_f = -(kT_e/2e) \ln m_i/2\pi m_e$ (using the Bohm sheath criterion).

— In the region $\Phi_B < 0$ ($\Phi_B < \Phi_p$ on fig.4.1), the current varies exponentially with the applied voltage, since $I_e \propto \exp(-e\Phi_B/kT_e)$. Plotting this part in log-log then makes it possible to determine robustly the electron temperature.

So, the characteristic makes it possible to determine independently and robustly determine the plasma temperature and the plasma density at infinity from the probe n_0.

4.4 Exercise and examples

4.4.1 Exercise : levitation of lunar dust

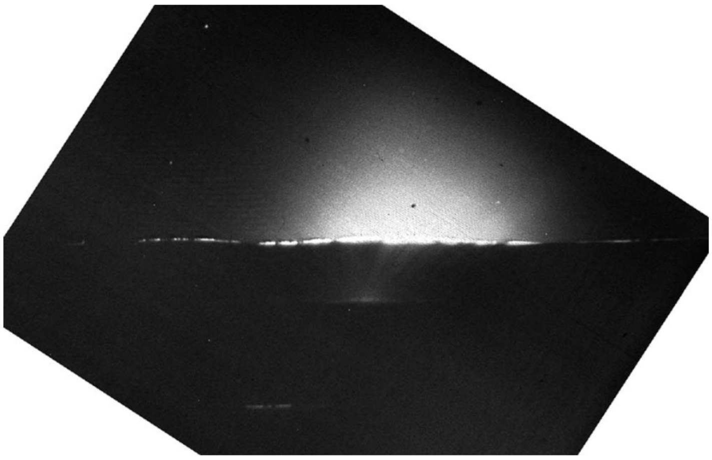

Fig. 1. Surveyor 6 image 328141526.354 showing a glow on the western lunar horizon after sunset. The broad and high diffuse glow is zodiacal light from interplanetary dust. The low bright band just at the horizon is lunar "Horizon Glow" apparently due to light scattered from dust particles near the lunar surface. National Space Science Data Center.

FIGURE 4.2 – From *Colwell et al, 2009*

When illuminated by the Sun, the lunar surface charges positively under the action of photoelectron emission. A photoelectron sheath is present above the surface. Its density is given by

$$n_{pe}(z) = n_{pe0}\left(1 + \frac{z}{\sqrt{2}\lambda_D}\right)^{-2} \tag{4.9}$$

where z is the altitude, n_{pe0} the density at the ground level, and λ_D is the photoelecton Debye length calculated with the density n_{pe0} and a temperature $T_{ph} \sim 3$ eV.

Calculate the altitude at which a dust grain levitates, as a function of its typical radius r.

4.5 The Bohm sheath criterion*

In chapter 2, we made a model of the plasma sheath next to a charged surface. Although practical to highlight the role of the Debye length, this model happened to not be completely accurate, because of an inappropriate modeling of the ions dynamics. Assume the object is charged negatively. The electrons or ions dynamics in a steady state is given by

$$n_\alpha u_\alpha \frac{du_\alpha}{dz} = kT_\alpha \frac{dn}{dz} - n_\alpha q_\alpha \frac{d\varphi}{dz} \tag{4.10}$$

The ratio of the the macroscopic kinetic energy term to the pressure term is of the order of the square of the Mach number $\mathrm{Ma}^2 = u^2/v_{th,\alpha}^2$, where the directed kinetic eneregy that a particle can acquire is of the order of the electrostatic potential drop $\varphi \sim kT_e$. For electrons, both terms are of the same order of magnitude – for the sake of simplicity, we will drop the kinetic energy term and assume that the electron density is given by the Boltzmann law

$$n_e(z) = n_\infty e^{e\varphi(z)/kT_e}. \tag{4.11}$$

On the other hand, because of the ion to electron mass ratio, the ion square mach number is by 3 orders of magnitude larger than the pressure term : our previous modeling of the ions as in Boltzmann equilibrium is not adequate, and we must instead calculate its density from the dynamics equation,

$$\frac{1}{2}m_i u_i(z)^2 + e\varphi(z) = \frac{1}{2}m_i u_i(\infty)^2 \Rightarrow u_i(z) = \sqrt{u_i(\infty)^2 - \frac{2e\varphi(z)}{m_i}} \tag{4.12}$$

together with the continuity equation

$$n_i(z)u_i(z) = n_\infty u_i(\infty) \Rightarrow n_i(z) = \frac{n_\infty}{\sqrt{1 - 2e\varphi(z)/(m_i u_i(\infty)^2)}}. \tag{4.13}$$

The potential in the sheath is thus given by the Poisson equation, but with the ion density given by eq.(4.14) instead of the Boltzmann formula. The resulting equation is strongly non-linear and no analytical solution can be found. Numerical solutions can be used for a proper modeling of the sheath. But in order to get a qualitative modeling of the sheath, we may just linearize the equation by assuming that $e\varphi \ll kT_e, m_i u_i(\infty)^2$. The linearized Poisson equation is

$$\frac{d^2\varphi}{dz^2} = \frac{1}{\lambda_{D,e}^2}\left(1 - \frac{kT_e}{m_i u_i(\infty)^2}\right)\varphi \tag{4.14}$$

where $\lambda_{D,e}^2 = \varepsilon_0 kT_e/n_\infty e^2$. If the parenthesis in the right hand side is negative, then we have for solution an harmonic oscillator : this is incompatible with our boundary conditions implying a steady plasma at infinity. So, the plasma must somehow organize itself so that the ions velocity at the entrance of the sheath verifies Bohm's criterion

$$u_i > \sqrt{\frac{kT_e}{m_i}}, \tag{4.15}$$

and the right hand side is sometimes called Bohm's velocity. In fact, the criterion is usually just fulfilled, and for practical cases it is possible to assume that ions practically enter the sheath with Bohm's speed.

Chapter 5: Motion of a charged particle in the elecromagnetic field

We have seen that a plasma is characterised by a strong coupling between the dynamics of the particles and those of the electromagnetic field. To understand a plasma, and in particular a magnetised plasma, it is therefore essential to have a good understanding of the motion of charged particles in given electromagnetic fields.

Throughout this section, $\mathbf{B} = B\mathbf{b}$ is the magnetic field vector. B is its modulus and $\mathbf{b}$ is the unit vector along the magnetic field line. The parallel component of a vector is $v_\| = \mathbf{v} \cdot \mathbf{b}$. Its perpendicular component is the remainder, $\mathbf{v}_\perp = \mathbf{v} - v_\| \mathbf{b}$.

5.1 Charged particle in constant fields

5.2 Particle in a constant and homogeneous field

We begin with the simplest case, where the magnetic and electric fields are constant in time and space. So here we have $B = const.$, and $\mathbf{b} = \mathbf{const}$.

5.2.1 Cyclotron motion

In the absence of an electric field, the Lorentz force acting on a particle of charge q and mass m is $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. This force does not do any work on the particle, since $\mathbf{F} \cdot \mathbf{v} = 0$. The kinetic energy of a particle in a purely magnetic field is therefore a constant of motion.

By separating the equation of motion $\dot{\mathbf{v}} = \mathbf{F}/m$ into parallel and perpendicular components, we obtain

$$\begin{cases} \dot{v}_\| = 0 \\ \dot{\mathbf{v}}_\perp = \omega_c \mathbf{v}_\perp \times \mathbf{b}. \end{cases} \tag{5.1}$$

where we have introduced the <u>*cyclotron frequency*</u> or <u>*gyro-frequency*</u> of the particle[1] $\omega_c = qB/m$. Thus, the parallel component of the particle's velocity is a constant of

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[1] Note that ω_c is an algebraic quantity, which can be positive or negative depending on the sign of q.

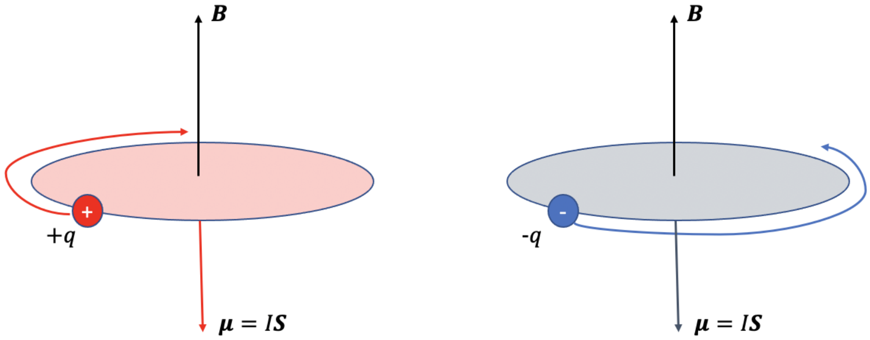

FIGURE 5.1 – Cyclotron motion of positively and negatively charged particles. The current $I \propto qv$ is oriented in the same direction in both cases, hence the antiparallel direction of the magnetic moment, independent of the particle's charge – see sec.5.2.2.

motion, and the norm of its perpendicular component is another. By introducing two Cartesian axes (x, y) in the plane perpendicular to $\mathbf{b}$, and using the complex notations $\bar{r}_\perp = x + iy$ and $\bar{v}_\perp = \dot{\bar{r}}_\perp = v_x + iv_y$, we obtain

$$\dot{\bar{v}}_\perp = -i\omega_c \bar{v}_\perp \Rightarrow \bar{v}_\perp = \bar{v}_\perp(0)e^{-i\omega_c t} \tag{5.2}$$

and the trajectory of the particle is given by

$$\dot{\bar{r}}_\perp = \bar{v}_\perp \Rightarrow \bar{r}_\perp = \bar{r}_\perp(0) + \frac{i\bar{v}_\perp(0)}{\omega_c}\left(e^{-i\omega_c t} - 1\right). \tag{5.3}$$

Thus, the particle describes a circle in the perpendicular plane. The radius of this circle is $\rho_\ell = |v_\perp/\omega_c|$, and is called the *Larmor radius* of the particle (and is a positive quantity). A positively charged particle describes a circle clockwise around $\mathbf{b}$ (right-handed polarisation), while a negatively charged particle describes a circle counterclockwise (left-handed polarisation).[2]

A little more vocabulary : the angle of the velocity vector relative to the magnetic field line is called *the pitch angle* θ ,

$$\theta = \arccos\frac{v_\parallel}{v} = \arctan\frac{v_\perp}{v_\parallel} \tag{5.4}$$

The phase of rotation of the particle in the perpendicular plane is called the *gyrophase*,

$$\varphi(t) = \omega_c t + \arg i\bar{v}_\perp(0) \tag{5.5}$$

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[2] The directions "clockwise" and "counterclockwise" are defined relative to a magnetic field pointing towards the observer.

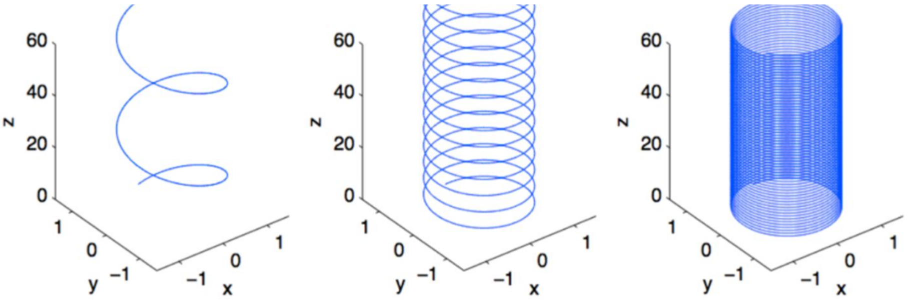

FIGURE 5.2 – Trajectories of a positively charged particle in a constant magnetic field directed along the z-axis, for three different values of its angle of attack ($\theta = 10°, 45°, 80°$ from left to right). Distances are normalised with respect to the particle's Larmor radius.

The average position (over the cyclotron motion) of a particle is called the position of its *guiding centre* $\mathbf{R}_g$. In this case,

$$\mathbf{R}_g(t) = \mathbf{R}_g(0) + v_{\parallel} t \mathbf{b}, \tag{5.6}$$

the guiding centre follows the straight magnetic field line at a constant speed.

In relativistic regime, the equation of motion is $\dot{\mathbf{p}} = q\mathbf{v} \times \mathbf{B}$ with $\mathbf{p} = \gamma m \mathbf{v}$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor. Taking the scalar product with $\mathbf{v}$, we immediately see that $v = const.$, so the Lorentz factor is also a constant and the above analysis remains valid, with the change $\omega_c = qB/\gamma m$. The Larmor radius is $\rho_\ell = \gamma m v_\perp / qB = p_\perp / qB$.

One last point of vocabulary : in the context of (astro)particle physics, the momentum of a particle in a magnetic field is often described by its *rigidity*, which is measured in volts and defined as follows :

$$R = \rho_\ell c B = \frac{p_\perp c}{|q|} \tag{5.7}$$

Useful orders of magnitude :
— Electron gyro-frequency

$$\omega_{c,e} \ [\text{rad.s}^{-1}] \simeq 176 \frac{B[\text{nT}]}{\gamma} \qquad f_{c,e} \ [\text{Hz}] \simeq 28 \frac{B[\text{nT}]}{\gamma} \tag{5.8}$$

— Ion gyro-frequency (number of charges Z, mass number A)

$$\omega_{c,i} \ [\text{rad.s}^{-1}] \simeq 10^{-1} \frac{Z}{A} \frac{B[\text{nT}]}{\gamma} \qquad f_{c,i} \ [\text{Hz}] \simeq 1.5 \times 10^{-2} \frac{Z}{A} \frac{B[\text{nT}]}{\gamma} \tag{5.9}$$

— Larmor radius (non-relativistic limit, $\mathscr{E}_\perp = p_\perp^2/2m = mv_\perp^2/2$)

$$\rho_{\ell,e}\ [\mathrm{km}] \simeq 3.4\frac{\sqrt{\mathscr{E}_\perp}\ [\mathrm{eV}]}{B\ [\mathrm{nT}]} \qquad \rho_{\ell,i}\ [\mathrm{km}] \simeq 144\frac{\sqrt{\mathscr{E}_\perp}\ [\mathrm{eV}]}{B\ [\mathrm{nT}]} \times \frac{\sqrt{A}}{Z} \tag{5.10}$$

where $\mathscr{E}_\perp$ is the perpendicular kinetic energy of the particle.

— Larmor radius (ultra-relativistic limit, $\mathscr{E}_\perp = p_\perp c$)

$$\rho_\ell\ [\mathrm{A.U.}] \simeq 2\times 10^{-2}\frac{\mathscr{E}_\perp\ [\mathrm{GeV}]}{B\ [\mathrm{nT}]} \equiv 2\times 10^{-2}\frac{R\ [\mathrm{GV}]}{B\ [\mathrm{nT}]} \tag{5.11}$$

5.2.2 Plasma diamagnetism

Charged particles in a field $\mathbf{B}_0$ rotate in such a way to produce a current that in turn generates a magnetic field $\delta\mathbf{B}$ that opposes $\mathbf{B}_0$. So, the plasma is a diamagnetic medium. The microscopic magnetic moment $\boldsymbol{\mu}$ associated to the current loop of a gyrating particle is

$$\boldsymbol{\mu} = I\mathbf{S} = -\frac{q\omega_c}{2\pi}\pi\rho_\ell^2\mathbf{b} = -\frac{\mathscr{E}_\perp}{B}\mathbf{b}. \tag{5.12}$$

It is independent of the particle's charge, since $q\omega_c > 0$ whatever is the sign of the particle. The magnetization vector of the plasma is the volumetric density of magnetic moment, which, in a plasma of density n, is

$$\mathbf{M} = 2n\boldsymbol{\mu} = -\frac{2nkT_\perp}{B}\mathbf{b}, \tag{5.13}$$

if we assume that the electrons and ions have the same temperature $kT_\perp = \langle\mathscr{E}_\perp\rangle$. Note that the plasma is not a linear medium.

Consider the following situation : a system of currents $\mathbf{j}_{ext}$, external to the plasma (for example circulating in a solenoid), produces a magnetic field $\mathbf{B}_0$ in the plasma. What decrease in the magnetic field inside the plasma will be produced by the plasma pressure ? Using Ampère's law, which in the magnetized medium, is

$$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{j}_{ext} + \nabla\times\mathbf{M}\right) = \nabla\times\left(\mathbf{B}_0 + \mu_0\mathbf{M}\right) \tag{5.14}$$

we have in the plasma

$$\mathbf{B} = \mathbf{B}_0 + \mu_0\mathbf{M} \simeq \left(1 - \frac{nkT_\perp}{B_0^2/2\mu_0}\right)\mathbf{B}_0 \simeq (1-\beta)\mathbf{B}_0 \tag{5.15}$$

where we have introduced the dimensionless *plasma* β parameter, equal to the ratio of the

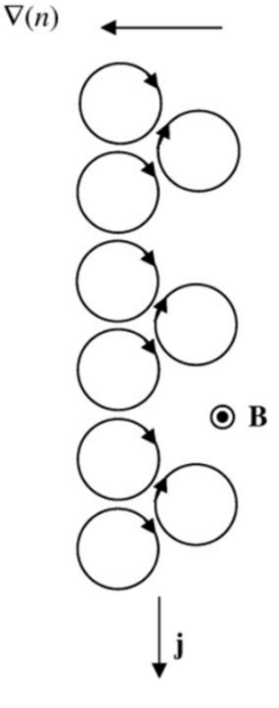

FIGURE 5.3 – Illustration of the origin of the plasma magnetisation current.

plasma pressure to the magnetic field pressure. Strictly speaking, our expression is valid only in the limit of very small values of β (since the magnetization has been calculated from the external field $\mathbf{B}_0$ and not self-consistently from the plasma magnetic field $\mathbf{B}$). This expression shows that the thermal pressure decreases the value of the external magnetic field inside the plasma.

The current density associated to this magnetization appearing in Ampère's law is

$$\mathbf{j}_{mag} = \nabla \times \mathbf{M} = -\frac{\nabla(2nkT_\perp) \times \mathbf{b}}{B} \tag{5.16}$$

and is called the magnetization current. It is perpendicular both to the magnetic field and to the pressure gradient. This current is not associated to a physical displacement of charged particles in the volume of the plasma, but results from the non-compensation of the currents carried by the Larmor rotation of the particles when a pressure gradient exists.

5.2.3 Constant electric field, crossed field drift

In the presence of a constant electric field, the equation of motion of the charged particle are now

$$\begin{cases} \dot{v}_\parallel = qE_\parallel/m \\ \dot{\mathbf{v}}_\perp = \omega_c \mathbf{v}_\perp \times \mathbf{b} + q\mathbf{E}_\perp/m. \end{cases} \tag{5.17}$$

Therefore, the motion along the magnetic field is uniformly accelerated, just as it would be in the absence of the magnetic field,

$$v_\parallel(t) = v_\parallel(0) + \frac{qE_\parallel t}{m} \qquad r_\parallel(t) = r_\parallel(0) + v_\parallel(0)t + \frac{qE_\parallel t^2}{2m} \tag{5.18}$$

The motion in the perpendicular plane consists of two components. The first is given by the solution of the homogeneous equation, and correspond to the cyclotron motion, as studied in the begining of this section – cf. eqs.(5.2)-(5.3) . The second component is given by a particular solution to the differential equation. A trivial solution is the one with constant velocity $\dot{\mathbf{v}}_{\perp,p} = 0$,

$$\mathbf{v}_{\perp,p} \times \mathbf{b} = -\frac{q\mathbf{E}_\perp}{m\omega_c} \Rightarrow \mathbf{v}_{\perp,p} = \frac{\mathbf{E}_\perp \times \mathbf{b}}{B}. \tag{5.19}$$

This constant perpendicular velocity appearing in the presence of an electric field is called the *E cross B drift*, or *crossed field drift* velocity. It plays a very important role in plasma physics, because it is the drift which maintains charged particles on a given field lines even when the latter are evolving in time.

$$\mathbf{v}_\times = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \tag{5.20}$$

Importantly, this drift does not depend on the charge nor on the mass of the particles : under the action of a constant electric field, the plasma drifts as a whole in the direction both perpendicular to **E** and **B**.

A direct and important interpretation of this drift comes the transformation of the electric field by a change of frame of reference. In the non-relativistic limit, the change is $\mathbf{E}' = \mathbf{E} + \mathbf{u}_{R'/R}\times\mathbf{B}$, while the magnetic field is invariant by a Galilean change of frame. Therefore, it is always possible to find a frame in which the electric field vanishes, and the motion of the charged particle consists in the cyclotron motion only. The velocity of this specific frame of reference is

$$\mathbf{u}_{R'/R}\times\mathbf{B} = -\mathbf{E} \Rightarrow \mathbf{u}_{R'/R,\perp} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = \mathbf{v}_\times \tag{5.21}$$

so, its perpendicular component is just the cross-field drift velocity (its parallel component is undetermined and can be anything). The velocity of the particle in the frame R is then the superposition of the cyclotron motion, which is the only motion in R', and the motion of the frame R' with respect to R. Therefore, one can think of the cross field velocity $\mathbf{v}_\times = \mathbf{u}_{R'/R}$ as the velocity of the magnetic field lines themselves. [3]

5.2.4 Constant force field

Under the action of a homogeneous force field **F**, the analysis carried out in the specific case of the electric field remains valid. Simply replace **E** with $\mathbf{F}/q$ to obtain the generic expression for the force drift

$$\mathbf{v}_F = \frac{\mathbf{F}\times\mathbf{B}}{qB^2}. \tag{5.22}$$

This force drift depends on the charge of the particle (if **F** does not depend linearly on q). Consequently, ions and electrons will generally drift in opposite directions, producing a drift current and plasma polarisation. Gravitational and inertial forces are, for these reasons, responsible for the appearance of plasma currents.

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[3] Consider a plasma in a solenoid. We start moving the solenoid at a velocity **V** with respect to an observer frame (lab. frame). An electric field appears in the solenoid in the lab frame, $\mathbf{E} = -\mathbf{V}\times\mathbf{B}$. What is the effect of this electric field on the plasma particles ?

5.3 Particle motion in an inhomogeneous field

In this section, we consider the motion of particles in fields that may vary in space, assuming weak variation on the length scale associated with cyclotron motion : $\rho_\ell \cdot \nabla \ll 1$

5.3.1 Movement of the guide centre : general equations

We separate the motion of a particle into a cyclotron component and the motion of the guide centre :

$$\mathbf{r}(t) = \mathbf{R}_g(t) + \mathbf{r}_\ell(t), \qquad \mathbf{v}(t) = \mathbf{V}_g(t) + \mathbf{v}_\ell(t), \tag{5.23}$$

where $\mathbf{v}_\ell(t) = \dot{\mathbf{r}}_\ell(t)$ is perpendicular to $\mathbf{b}$ and is the solution to the equation

$$\frac{d\mathbf{v}_\ell}{dt} = \frac{q}{m}\mathbf{v}_\ell \times \mathbf{B}(\mathbf{R}_g), \tag{5.24}$$

Averaging these expressions over the cyclotron motion gives $\langle \mathbf{r}(t) \rangle = \mathbf{R}_g(t)$, and $\langle \mathbf{v}(t) \rangle = \mathbf{V}_g(t)$.

We perform a Taylor expansion of the fields on the Larmor rotation scale of the particles

$$\mathbf{E}(\mathbf{r}, t) \simeq \mathbf{E}(\mathbf{R}_g) + \mathbf{r}_\ell \cdot \nabla \mathbf{E}(\mathbf{R}_g) \qquad \text{and } \mathbf{B}(\mathbf{r}, t) \simeq \mathbf{B}(\mathbf{R}_g) + \mathbf{r}_\ell \cdot \nabla \mathbf{B}(\mathbf{R}_g). \tag{5.25}$$

The dynamics of the particle in the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ are now described by

$$\frac{d\mathbf{V}_g}{dt} + \frac{d\mathbf{v}_\ell}{dt} = \frac{q}{m}\left(\mathbf{E}(\mathbf{R}_g) + \mathbf{r}_\ell \cdot \nabla \mathbf{E}(\mathbf{R}_g) + (\mathbf{V}_g + \mathbf{v}_\ell) \times (\mathbf{B}(\mathbf{R}_g) + \mathbf{r}_\ell \cdot \nabla \mathbf{B}(\mathbf{R}_g))\right). \tag{5.26}$$

Using definition (5.24) and averaging over the cyclotron motion (which cancels out all linear terms in $\mathbf{r}_\ell$), we obtain

$$m\frac{d\mathbf{V}_g}{dt} = q\left(\mathbf{E}(\mathbf{R}_g) + \mathbf{V}_g \times \mathbf{B}(\mathbf{R}_g) + \langle \mathbf{v}_\ell \times \mathbf{r}_\ell \cdot \nabla \mathbf{B}(\mathbf{R}_g) \rangle\right). \tag{5.27}$$

The term in brackets can be evaluated in a somewhat tedious manner (see exercise), in the form

$$\langle \mathbf{v}_\ell \times \mathbf{r}_\ell \cdot \nabla \mathbf{B}(\mathbf{R}_g) \rangle = -\frac{v_\perp^2}{2\omega_c}\nabla B = -\frac{\mu}{q}\nabla B. \tag{5.28}$$

It is proportional to the gradient of the modulus of the magnetic field. The equation of motion of the particle's guide centre is thus

$$m\frac{d\mathbf{V}_g}{dt} = q\left(\mathbf{E}(\mathbf{R}_g) + \mathbf{V}_g \times \mathbf{B}(\mathbf{R}_g)\right) - \mu \nabla B. \tag{5.29}$$

We now assume that the solution to this equation can be written as [4]

$$\mathbf{V}_g(t) = v_\parallel(t)\mathbf{b} + \mathbf{v}_\times + \mathbf{v}_\perp^1 \tag{5.30}$$

where $\mathbf{v}_\times = \mathbf{E}(\mathbf{R}_g) \times \mathbf{B}(\mathbf{R}_g)/\mathbf{B}(\mathbf{R}_g)^2$, and $d\mathbf{v}_\perp^1/dt \ll d\mathbf{v}_\times/dt$. This amounts to decomposing the motion into parallel and perpendicular components on the one hand, and decomposing the perpendicular component into a cross-field drift component, which would be present even in the absence of any field inhomogeneity, and a slowly varying component $\mathbf{v}_\perp^1$ linked to the existence of inhomogeneities.

The derivative of V_g is written (neglecting the derivative of the term $\mathbf{v}_\perp^1$)

$$\frac{d\mathbf{V}_g}{dt} \simeq \frac{dv_\parallel}{dt}\mathbf{b} + v_\parallel\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{v}_\times}{dt} \tag{5.31}$$

This ultimately allows us to write the equation of motion in the form

$$m\left(\frac{dv_\parallel}{dt}\mathbf{b} + v_\parallel\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{v}_\times}{dt}\right) = q\mathbf{E}_\parallel + q\mathbf{v}_\perp^1 \times \mathbf{B}(\mathbf{R}_g) - \mu\nabla B. \tag{5.32}$$

This last equation forms the basis for studying the motion of the guide centre in non-homogeneous fields. Note that the inhomogeneity of the electric field plays no role in this motion [5]. In the following paragraph, we study parallel motion, then we will examine perpendicular drifts.

5.3.2 Mirror force and conservation of magnetic moment

Taking the scalar product of equation (5.32) with $\mathbf{b}$, we obtain :

$$m\frac{dv_\parallel}{dt} = qE_\parallel - \mu\mathbf{b}\cdot\boldsymbol{\nabla}B \tag{5.33}$$

There are two terms on the right-hand side. On the one hand, there is the electric field that accelerates the particle along the field line. On the other hand, there is a term called the *mirror force*, which we will see acts as a pseudo-force that modifies the angle of the velocity vector with the average field, in the presence of a parallel magnetic field gradient (converging or diverging field lines).

In what follows, we will only consider the mirror force term. Let us show that equation (5.33) implies that the magnetic moment μ of the particle is a constant of motion. The variation in parallel kinetic energy is

$$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu v_\parallel\mathbf{b}\cdot\boldsymbol{\nabla}B = -\mu\frac{dB}{dt} \tag{5.34}$$

---

[4] This is a convenient way of obtaining the correct result, although maybe not the most rigorous. The perturbative treatment in the frame of Hamiltonian formalism can be found in the literature, but is rather cumbersome.

[5] In reality, it intervenes as a second-order effect, involving the second derivative of $\mathbf{E}(\mathbf{r})$

where we have assumed that the magnetic field does not vary with time ($\partial_t B = 0$) and $dB/dt = \mathbf{v}\cdot\nabla B$ is the variation of the field along the particle's trajectory. Since the effect does not involve any electric field, the kinetic energy of the particle remains constant (the magnetic force does not do any work). We must therefore have

$$\frac{d}{dt}\left(\mathscr{E}_\perp + \mathscr{E}_\parallel\right) = \frac{d}{dt}\left(\mu B\right) - \mu\frac{dB}{dt} = B\frac{d\mu}{dt} = 0. \tag{5.35}$$

Since $B \neq 0$, the magnetic moment μ must be conserved along the particle's trajectory.

In the absence of an electric field (or any other force exerting work on the particle), we can rewrite the conservation of the particle's kinetic energy in the following form :

$$\frac{1}{2}mv_\parallel^2(s) + \mu B(s) = \mathscr{E} = const. \tag{5.36}$$

where s is a coordinate along a magnetic field line. Since μ is a constant, the modulus of the magnetic field $B(s)$ acts exactly like potential energy for parallel motion : a large increase in the magnetic field (corresponding to a strong convergence of the field lines) will reflect the charged particles like a mirror, and a "hole" in the magnetic field can trap charged particles like a potential well. Such a trapping magnetic field configuration is usually called a magnetic bottle.

Magnetic bottle

A magnetic field configuration with two points of convergence of the lines is called a magnetic bottle. Such a configuration is characterised by the ratio $R_m = B_{max}/B_{min}$ (*mirror ratio*), which characterises its efficiency in trapping particles.

A particle will escape from the bottle if its total kinetic energy satisfies $\mathscr{E} > \mu B_{max}$. If we call θ_{min} the pitch angle of the particle at the position where the magnetic field is B_{min}, then we can write the escape condition as follows

$$\sin^2\theta_{min} < \frac{B_{min}}{B_{max}} = 1/R_m. \tag{5.37}$$

The expression (5.37) defines $\theta_m = \arcsin\sqrt{1/R_m}$, the half-angle at the apex of a cone in velocity space, called the *loss cone.* Particles inside the loss cone will be able to escape from the magnetic trap. Consequently, the velocity distribution function of the plasma inside a magnetic bottle is generally not an isotropic Maxwellian function at equilibrium, but a function emptied of particles with small pitch-angles (if collisions are completely neglected). Collisions (or certain electromagnetic instabilities) will tend to send particles inside the loss cone, producing a continuous leak—magnetic bottles are never really plasmatight.

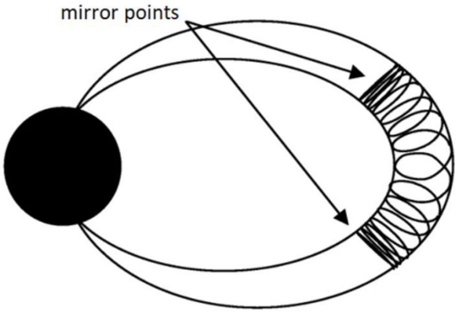

FIGURE 5.4 – Trajectory of a particle in a magnetic bottle configuration produced by the converging field lines of a planetary magnetic dipole.

Conservation of magnetic flux through the Larmor contour

One consequence of the mirror force is that the magnetic field flux through a surface lying on the contour defined by the particle's Larmor radius is always conserved. This can be easily shown from the expression for this flux

$$\Phi_B = \iint \mathbf{B} \cdot d\mathbf{S} \simeq B(s)\pi\rho(s)^2 = \frac{\pi m^2}{q^2}\frac{v_\perp^2}{B} = \frac{2\pi m}{q^2}\mu = const. \tag{5.38}$$

This result is particularly useful for visualising the trajectory of a charged particle : the trajectory winds around a magnetic flux tube. This is illustrated in Figure 5.4.

5.3.3 Perpendicular drifts

We obtain the perpendicular motion of the guide centre by taking the vector product of equation (5.32) by **b**. We obtain

$$\mathbf{v}_\perp^1 = \frac{1}{\omega_c}\mathbf{b} \times \left(\frac{\mu}{m}\boldsymbol{\nabla} B + v_\parallel \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{v}_\times}{dt}\right) \tag{5.39}$$

The first term in parentheses describes the drift known as the *grad B drift* (unsurprisingly). The other two terms describe inertial effects related to the existence of acceleration of the guide centre. The first of these two terms is the curvature drift, and the last is the polarisation drift.

Note that none of the terms in parentheses depend on the charge or mass of the particle. Thus, the only dependencies on mass and charge are contained in ω_c. This has two consequences : first, electrons and ions will drift in opposite directions in an inhomogeneous B field, creating polarisation currents in the plasma. Second, all these drift velocities are proportional to the mass-to-charge ratio and will therefore be carried mainly by the ionic species in the plasma. We review these three terms in the following paragraphs.

Grad-B drift

Grad-B drift results from the existence of a perpendicular gradient of the magnetic field modulus. The expression for the drift velocity is as follows :

$$\mathbf{v}_\nabla = \frac{\mu}{q}\frac{\mathbf{B}\times\nabla B}{B^2} = -\frac{mv_\perp^2}{2qB}\frac{\nabla B\times\mathbf{B}}{B^2} \tag{5.40}$$

The guide centre drifts in a direction perpendicular to both the field and the field gradient. Qualitatively, this can be interpreted as the Larmor radius being slightly smaller in the high-field region than in the low-field region, resulting in a cycloidal trajectory in the direction perpendicular to ∇B.

Curvature drift

The total derivative of $\mathbf{b}$ along the particle's trajectory is

$$\frac{d\mathbf{b}}{dt} = \partial_t\mathbf{b} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{b} = \partial_t\mathbf{b} + \mathbf{v}_\times\cdot\boldsymbol{\nabla}\mathbf{b} + v_\parallel\mathbf{b}\cdot\boldsymbol{\nabla}\mathbf{b}, \tag{5.41}$$

as a result of which this inertial drift is strictly speaking composed of three terms. In practice, we almost always have $v_\parallel \gg v_\times$ and $v_\parallel\mathbf{b}\cdot\nabla \gg \partial_t$ (i.e., the particle will perceive spatial changes in the direction of $\mathbf{B}$ along its trajectory much more quickly than any intrinsic temporal changes in the direction of the field line). The drift is thus generally almost entirely due to the curvature term, hence its name. Introducing the local radius of curvature of the field line R_c such that

$$\mathbf{b}\cdot\nabla\mathbf{b} = -\frac{\mathbf{n}}{R_c} \tag{5.42}$$

where $\mathbf{n}$ is the unit vector perpendicular to the trajectory (pointing outwards from the centre of curvature), we can conveniently express the drift as follows

$$\mathbf{v}_c = \frac{m}{q}\frac{\mathbf{B}\times v_\parallel^2(\mathbf{b}\cdot\nabla\mathbf{b})}{B^2} = \frac{mv_\parallel^2}{qR_c}\frac{\mathbf{n}\times\mathbf{B}}{B^2} \tag{5.43}$$

Finally, note that the left-hand side of the vector product is the centrifugal force $\mathbf{F} = mv_\parallel^2\mathbf{n}/R_c$ applied to a particle following a curved trajectory at constant velocity $v_\parallel$. This curvature drift then appears as a special case of force drift discussed previously – eq.(5.22), for the centrifugal force.

Polarisation drift

The last term of equation (5.39) involves the time derivative of the cross-field drift. It is particularly important when there is a time-varying perpendicular electric field in the plasma, $\dot{\mathbf{E}}_\perp$. The drift velocity in this case is

$$\mathbf{v}_p = \frac{m}{qB^2}\frac{\mathbf{B}\times(\dot{\mathbf{E}}_\perp\times\mathbf{B})}{B^2} = \frac{m}{qB^2}\frac{d\mathbf{E}_\perp}{dt}, \tag{5.44}$$

and is called the polarisation drift. This drift is (for once) parallel to the applied electric field and contributes significantly to the perpendicular polarizability (or dielectric response) of a magnetised plasma at low frequency ($\omega \ll \omega_c$), hence its name. The polarisation current produced by applying an alternating electric field to a plasma is (neglecting the contribution of electrons) :

$$\mathbf{j}_p = \sum_s n_s q_s \mathbf{v}_s \simeq \frac{nm_i}{B^2}\frac{d\mathbf{E}_\perp}{dt} = \frac{\partial \mathbf{P}}{\partial t} \Rightarrow \mathbf{P} \simeq \frac{nm_i}{B^2}\mathbf{E}_\perp \equiv \varepsilon_0 \chi_\perp \mathbf{E}_\perp \tag{5.45}$$

where $\mathbf{P}$ is the plasma polarisation vector, $\chi_\perp = \rho_m/\varepsilon_0 B^2$ is the dielectric susceptibility and $\varepsilon_\perp = \varepsilon_0(1+\chi_\perp)$ is the perpendicular dielectric function (i.e. for an applied field $\mathbf{E}_\perp$ perpendicular to the magnetic field) of the magnetised plasma. We have introduced the plasma density $\rho_m \simeq nm_i$ and assumed ions with a charge of $+e$.

5.4 Adiabatic invariants

Here we recall a useful result from analytical mechanics. A periodic motion with period T is characterised by the existence of invariants known as *Poincaré invariants*, which remain approximately constant under variations in the system parameters that are slow relative to T. These invariants take the form

$$I = \oint \mathbf{p} \cdot d\mathbf{q} \tag{5.46}$$

where $\mathbf{p}$ and $\mathbf{q}$ are conjugate dynamic variables, and the closed integral implies that it is performed over a complete period, whose trajectory effectively describes a closed curve in phase space. These invariants can also be formulated, in a way that is often more practical in applications, as an integral over time

$$I = \int_{T_q} W_q(t)dt = \langle W_q(t)\rangle T_q, \tag{5.47}$$

where T_q is the period associated with the periodic motion of the coordinate q, and W_q is the energy associated with the degree of freedom described by (p, q).

It can be shown that these integrals are conserved to first order in T_q/τ, where τ is the timescale over which the perturbation of the system is applied.

The existence of these invariants proves very useful for studying periodic motions in general, and the motion of charged particles in a magnetic field in particular. The example of the trajectory of a particle in a magnetic bottle whose structure varies slowly over time (or, similarly, of a particle trapped in the Earth's magnetic dipole field) is interesting and "paradigmatic" in this respect.

5.4.1 First adiabatic invariant : the magnetic moment

We first consider the fastest periodic motion of our system, i.e. the cyclotron motion. The associated adiabatic invariant is

$$I_1 = \langle W_q(t) \rangle T_q = \frac{1}{2} m v_\perp^2 \frac{2\pi}{\omega_c} = \frac{2\pi m}{q} \mu \tag{5.48}$$

which is just the magnetic moment of the particle, to constant factor. We recover the constancy of this quantity, that we've demonstrated through the mirror force, for a time varying magnetic field.

5.4.2 Second adiabatic invariant : bounce motion

In a magnetic trap, the particle will oscillate between two mirror points defined by $\mu B(s_m) = \mathscr{E}$, as seen previously. Let's call T_b the "bounce period" associated to this motion. The associated adiabatic invariant is

$$I_2 = \langle W_s(t) \rangle T_s = \frac{1}{2} m \langle v_\parallel^2 \rangle T_b \tag{5.49}$$

So if the bouce period of the particle is varying in time (on timescales much larger than T_b), the parallel kinetic energy of the particle will vary as well. There are two main reasons why it may occur :

— Perpendicular drifts may convect to shorter field lines (closer mirror points) : then the Bounce period decreases and the mean kinetic energy of the particle will increase. Of course the opposite reasoning applies if the particle drift toward longer field lines.

— The magnetic field configuration may have some intrinsic time variation. For example the magnetic bottle may contract on itself, and the particle will be trapped between two approaching magnetic walls. It will as a consequence gain energy : this phenomena is called the first-order Fermi acceleration, and can be responsible for the production of cosmic rays of very high energies.

5.4.3 Third adiabatic invariant : enclosed magnetic flux

A third periodic motion may be identified in magnetic traps : once averaged over the bounce motion, the particle may be seen as a "magnetic shell", consisting in the magnetic flux tube bounded by its two mirror points. This magnetic shell moves perpendicularly to the field lines under the effect of the perpendicular drifts. In the example of a particle trapped in the Earth dipolar field, this motion will be azimuthal and associated to a momentum $p_\phi = m v_\phi + q A_\phi$, where $\mathbf{A}$ is the field vector potential. Then

$$I_3 = \int_0^{2\pi} p_\phi r d\phi \simeq q \oint \mathbf{A} \cdot d\boldsymbol{\ell} = q \Phi_B \sim q \pi R^2 B_0 \tag{5.50}$$

where Φ_B is the total magnetic flux enclosed by the azimuthal motion of the particle around the earth, and R the approximate radius of the particle's orbit. Thus, if, for some reason, the effective magnetic field B_0 of the Earth increases, the particle orbit will tend to diminish its radius to keep the enclosed flux constant. This invariant is not in practice very useful, because events making the total magnetic field vary (e.g. magnetic storms caused by the interaction of the magnetosphere with a coronal mass ejection) will tend to occur on timescales that are of the order, or smaller, than the periodic motion of particles around the Earth, and the adiabatic invariant is not conserved under these non-adiabatic conditions.

5.5 Examples and exercises

5.5.1 The term $\mu\nabla\mathbf{B}$

In the course, we assumed that the cyclotron period averaging term

$$m = \langle \mathbf{v}_\ell \times \mathbf{r}_\ell \cdot \boldsymbol{\nabla}\mathbf{B}(\mathbf{R}_g)\rangle = -\frac{\mu}{q}\boldsymbol{\nabla}B. \tag{5.51}$$

In this exercise, we will demonstrate this. To do so, we will proceed in several steps

1. First, show that
$$m = \omega_c\left((\langle \mathbf{r}_\ell\mathbf{r}_\ell\rangle : \boldsymbol{\nabla}\mathbf{B})\mathbf{b} - \langle \mathbf{r}_\ell\mathbf{r}_\ell\rangle \cdot \boldsymbol{\nabla}B\right) \tag{5.52}$$
Hint : (i) What connection does the expression for the cyclotron trajectory allow us to make between $\mathbf{r}_\ell$ and $\mathbf{v}_\ell$? (ii) Expressing scalar/tensor products in Einstein notation helps to find the desired forms. (iii) We will assume that the direction of the vector $\mathbf{b}$ is constant "during the time it takes to calculate the average".
2. Express the tensor $\mathbf{r}_\ell\mathbf{r}_\ell$ as a matrix in a Cartesian basis with axis $\mathbf{z} \equiv \mathbf{b}$. Averaging over the gyrophase to show that
$$\langle \mathbf{r}_\ell\mathbf{r}_\ell\rangle = \frac{v_\perp^2}{2\omega_c^2}\left(\mathbf{I} - \mathbf{bb}\right) \tag{5.53}$$
3. Combining the results of the two previous questions (and a suitably chosen Maxwell equation), show that $m = -\frac{\mu}{q}\boldsymbol{\nabla}B$.

5.5.2 Particle orbit on a potential field line.

We have studied the effect of magnetic field line curvature. However, Maxwell's equations do not allow for the existence of field line curvature without a gradient in the associated field modulus (because div $\mathbf{B} = 0$) ; therefore, a grad-B drift always accompanies a curvature drift. In this exercise, we seek to calculate the drift in the case of a potential field (i.e. no current density or displacement current in the plasma : **rot B** $= 0$), characterised by a constant radius of curvature R_c.

1. Consider a potential field line with a radius of curvature R_c. Calculate the perpendicular gradient $\boldsymbol{\nabla} B$ of the field at a point on this field line (for this, it may be convenient to use a polar coordinate system).
2. Deduce the expression for the perpendicular drift of a particle moving along this field line, showing the total kinetic energy of the particle $\mathscr{E}$ and its parallel energy $\mathscr{E}_{\parallel}$.

5.5.3 Time invariance of the magnetic moment

We have shown the invariance of the magnetic moment μ along a particle trajectory if the magnetic field is slowly varying in space. Let's show that it is also invariant if $\mathbf{B}$ is homogeneous but slowly varying in time.

If $\partial_t \mathbf{B} \neq 0$, there will be an electric field associated to this time variation, and a change of the kinetic energy $\mathscr{E}_\perp$ of the particle due to the work of this electric force along its cyclotron trajectory. Its small variation during a cyclotron cycle is

$$\delta \mathscr{E}_\perp = \oint q\mathbf{E} \cdot \mathbf{v}_\perp dt = -|q| \oint \mathbf{E} \cdot d\boldsymbol{\ell} = |q| \iint \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{5.54}$$

where we used the Stokes theorem and Faraday's law to get the last part of the equality, and oriented $d\ell$ in the anti-clockwise direction, consistently with the Stokes law.

Now we assume that the magnetic field is *slowly varying*, that is, its variation on a gyroperiod is a small quantity δB. Then to a good approximation, one has

$$\delta \mathscr{E}_\perp = |q| \frac{\delta B}{\delta t} \iint dS = \frac{q\omega_c \delta B}{2\pi} \pi \rho^2 = \mathscr{E}_\perp \frac{\delta B}{B} \tag{5.55}$$

from which we finally obtain that the small variation

$$\delta \left(\frac{\mathscr{E}_\perp}{B} \right) = \delta \mu = 0. \tag{5.56}$$

The magnetic moment of the particle is, as a consequence, approximately invariant for slow variations of the magnetic field.

Chapter 6: Elements of kinetic theory

In this chapter, we introduce the most fundamental description of a plasma : the kinetic description, which aims to describes, etymologically speaking, the motion (κινητικός) of the particles of the plasma. Since there are a lots of them, the kinetic treatment will involve an important reduction of the amount of information available, usually down to a single particle phase space distribution function. We will first derive the equations which describe the time-evolution of this phase space distribution, then see how this description connects to the fluid description of plasmas. Finally we investigate the linear behaviour of an electron plasma in the kinetic description, and introduce the important notion of Landau damping of plasma oscillations.

6.1 Time-evolution of the phase space distribution function

In this section we introduce the phase space distribution function, or phase space density $f(\mathbf{r}, \mathbf{v}, t)$, which counts the number of particles in a particular volume of phase space : $dN = f(\mathbf{r}, \mathbf{v}, t)d^3\mathbf{r}d^3\mathbf{v}$, we differentiate between a microscopic distribution containing full information on the system, and a mesoscopic distribution averaged over small over phase space volumes – and derive time evolution equations for both.

6.1.1 Microscopic dynamics

We assume that a particle of the specie s, labelled p, evolves in phase space according to the laws of motion

$$\frac{d\mathbf{r}_p}{dt} = \mathbf{v}_p \tag{6.1}$$

$$\frac{d\mathbf{v}_p}{dt} = \frac{\mathbf{F}(\mathbf{r}_p, \mathbf{v}_p)}{m_s} \tag{6.2}$$

and we make the supplementary assumption that the force field is divergent-free in velocity space[1], that is

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{F}(\mathbf{r}, \mathbf{v}) = 0. \tag{6.3}$$

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[1] You may check that this is valid for the Lorentz force, using $\nabla \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\nabla \times \mathbf{x}) - \mathbf{x} \cdot (\nabla \times \mathbf{y})$

The system of N particles is thus entirely described by $6N$ first order differential equations, or 3N second order differential equations. Given initial conditions, the solution of the differential system gives us $6N$ trajectories, so that we can follow the evolution of the system in phase space. In the following, we seek to replace this description by a Eulerian description of phase space : we shall consider a phase space volume around coordinates $(\mathbf{r}, \mathbf{v})$, and seek for an evolution equation of the distribution function at this particular location.

6.1.2 Klimontovitch's equation

We introduce the distribution of N_s point particles of specie s in a 6 dimensional phase space[2],

$$\tilde{f}_s(\mathbf{r}, \mathbf{v}, t) = \sum_{p=1}^{N_s} \delta(\mathbf{v} - \mathbf{v}_p(t))\delta(\mathbf{r} - \mathbf{r}_p(t)) \tag{6.4}$$

which is sometimes called the Klimontovitch distribution function. This function contains all the information there is to know about the system of point particles, and its evolution according to the laws of physics is completely deterministic (no loss of information) and given by eqs.(6.2). The integral of this distribution on a small phase space volume $d^3\mathbf{v}d^3\mathbf{r}$ gives the number of particles in this volume, and the integral of the distribution on velocity space gives the number density of particles of specie s,

$$\tilde{n}_s(\mathbf{r}, t) = \int \tilde{f}_s(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{v} = \sum_{p=1}^{N_s} \delta(\mathbf{r} - \mathbf{r}_p(t)). \tag{6.5}$$

This density (and the associated current) can be used in principle to calculate the fields $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ and to close the microscopic system of equations. The number of particles in a plasma (say, of the order of the Avogadro number $N_A \sim 10^{23}$), however, makes it in practice impossible to use. This justifies the introduction of a smoothed distribution function, which includes a reduced information on the system and will be more convenient to use in practice.

It is important to highlight the interpretation to be given to the *ensemble average* of the Klimontovitch distribution. For this consider a random variable $\mathbf{X}$ distributed according to the probability density $\rho(\mathbf{X})$ (i.e. the probability that a realization of $\mathbf{X}$ falls between $\mathbf{X}$ and $\mathbf{X} + d\mathbf{X}$ is $dp = \rho(\mathbf{X})d\mathbf{X}$)). The expectation value of the Dirac distribution $\delta(\mathbf{x} - \mathbf{X})$ is

$$\langle \delta(\mathbf{x} - \mathbf{X}) \rangle = \int \rho(\mathbf{X})\delta(\mathbf{x} - \mathbf{X}) d\mathbf{X} = \rho(x).$$

Therefore, considering in the Klimontovitch equation the N_s particles as uncorrelated random variables, distributed according to the probability density $f_1(\mathbf{r}, \mathbf{v}, t)$, one has

$$\langle \tilde{f}_s(\mathbf{r}, \mathbf{v}, t) \rangle = N_s f_1(\mathbf{r}, \mathbf{v}, t)$$

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[2] The $\tilde{x}$ (tilde) symbol is used to denote the microscopic, fastly variable, quantities, by opposition to the smoothed quantities that will be introduced in the following of the section

In this expression $f_1(\mathbf{r}, \mathbf{v}, t)$ is a smoothed distribution function, known as the 1-particle probability distribution function. It gives the probability to find a randomly chosen particle in a phase space element $d^3\mathbf{r}d^3\mathbf{v}$, at a time t.

Let's now look for an evolution equation for the microscopic distribution $\tilde{f}_s(\mathbf{r}, \mathbf{v}, t)$. For this we compute its time derivative (sommation on the index p is assumed in all the following) :

$$\frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} = -\mathbf{v}\delta(\mathbf{v} - \mathbf{v}_p)\frac{\partial}{\partial \mathbf{r}}\delta(\mathbf{r} - \mathbf{r}_p) - \frac{\mathbf{F}(\mathbf{r}, \mathbf{v})}{m_s}\delta(\mathbf{r} - \mathbf{r}_p)\frac{\partial}{\partial \mathbf{v}}\delta(\mathbf{v} - \mathbf{v}_p) \tag{6.6}$$

where we expanded the derivative of the product of two functions, and used the property of the delta that for any function g, $g(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) = g(\mathbf{y})\delta(\mathbf{x} - \mathbf{y})$.

$$\frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} = -\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\delta(\mathbf{r} - \mathbf{r}_p)\delta(\mathbf{v} - \mathbf{v}_p) - \frac{\mathbf{F}(\mathbf{r}, \mathbf{v})}{m_s}\frac{\partial}{\partial \mathbf{v}}\delta(\mathbf{v} - \mathbf{v}_p)\delta(\mathbf{r} - \mathbf{r}_p) \tag{6.7}$$

so that

$$\frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{r}} + \frac{\mathbf{F}(\mathbf{r}, \mathbf{v})}{m_s} \cdot \frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}} = 0 \tag{6.8}$$

which gives the time evolution of the distribution $\tilde{f}_s$ and is usually named Klimontovitch's equation. This is a bit formal, but has the advantage of encompassing all the dynamics of the N_s particles into a single equation.

If the force field is divergence free in velocity space (as hypothesized in the previous section) this equation can be cast in the conservative form

$$\frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{v}\tilde{f}_s(\mathbf{r}, \mathbf{v}, t) + \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\mathbf{F}(\mathbf{r}, \mathbf{v})}{m_s}\tilde{f}_s(\mathbf{r}, \mathbf{v}, t) = 0, \tag{6.9}$$

or

$$\frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \boldsymbol{\nabla}_6 \cdot \mathbf{U}_6\tilde{f}_s(\mathbf{r}, \mathbf{v}, t) = 0, \tag{6.10}$$

which shows that Klimontovitch's equation is a conservation equation in phase space, associated to a flow of 6-velocity $\mathbf{U}_6 \equiv (\mathbf{v}, \mathbf{F}/m_s)$.

6.1.3 Mean-field dynamics

Klimontovitch distribution captures a lot of information, and we need to drop a lot of it in order to reach a practically usable formalism. A straightforward, and mathematically clean way to do so is to look for an equation on the ensemble average $\langle\tilde{f}_s\rangle = f_{1,s} \equiv f_s(\mathbf{r}, \mathbf{v}, t)$. This is what we do in the following – but as a first step, and since the ensemble average procedure is not always intuitive, we introduce the more intuitive "coarse graining" average of the Klimontovitch distribution on a mesoscopic scale.

The existence of a mesoscopic scale

In the fashion of fluid dynamics, we introduce a mesoscopic volume in phase space $d^3\mathbf{v}d^3\mathbf{r}$, containing enough particles for averages to make sense, and average the distribution over it. The averaging procedure can be thought of as counting the number of particles ΔN_s in a phase space volume $\Delta V_v \Delta V_r$, dividing by the result by the phase space volume to get a smoothed distribution $f(\mathbf{r}, \mathbf{v}, t)$,

$$f_s(\mathbf{r}, \mathbf{v}, t) \simeq \frac{\int_{\Delta V_r, \Delta V_v} \tilde{f}_s(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{v} d^3\mathbf{r}}{\Delta V_r \Delta V_v} \tag{6.11}$$

This average makes sense only if the r.m.s fluctuations δf_s are small compared to the expectation value $f_s(\mathbf{r}, \mathbf{v}, t)$. The ratio $\delta f_s / f_s \sim 1/\sqrt{\Delta N_s}$, so one must have a large number of particles in any mesoscopic phase space volume.

The order of magnitude of the distribution function, at least around its "thermal bulk" is $f \sim N/(L^3 v_T^3) \sim 1/(\ell^3 v_T^3)$, where L is the size of the system, ℓ is the mean interparticular distance and v_T the thermal speed. We want $f \Delta V_v \Delta V \gg 1$, so

$$\Delta r \Delta v / \ell v_T \gg 1$$

So, we need our volume large enough to contain lots of particles. On the other hand we want it small enough to resolve the physical scales of the system, typically the Debye length and the thermal speed. The question is : does such an intermediate scale exist ?

Let's take for the spatial extension of our mesoscopic element $\Delta r = \alpha \lambda_D = \alpha \Gamma^{-1/3} \ell$ and $\Delta v = \beta v_T$. The condition to have a large number of particles in the phase space mesoscopic volume now reads $\alpha\beta \gg \Gamma^{1/3}$. On the other hand we want to keep Δr and Δv both small compared to typical length and velocity scales, to resolve the variations of the distribution function, so

$$1 \gg \alpha\beta \gg \Gamma^{1/3}, \qquad \alpha \ll 1, \quad \beta \ll 1$$

is the condition on α and β. We can choose $\alpha \sim \beta \sim \Gamma^{1/12} \ll 1$. So the smallness of the plasma parameter Γ determined the possibility to build the mesoscopic scale that we are looking for. In typical astrophysical or laboratory plasmas, $\Gamma^{1/12} \sim 1/7$, which makes the kinetic treatment relevant but does not give a huge margin either. It must be kept in mind that in rarefied regions of phase space, for instance for velocities of a few thermal velocities, the results provided by a mean-field treatment must be carefully questioned.

Evolution of the smooth distribution function

Now that we have constructed our phase-space averaging scheme, we introduce the bracket notation to denote the averaging over phase space mesoscopic volumes – or equivalently[3] ensemble average,

$$f_s(\mathbf{r},\mathbf{v},t) \equiv \langle \tilde{f}_s(\mathbf{r},\mathbf{v},t)\rangle. \tag{6.12}$$

We define the smooth field, or *mean-fields* as $\mathbf{E} = \langle\tilde{\mathbf{E}}\rangle$, and $\mathbf{B} = \langle\tilde{\mathbf{B}}\rangle$. We now introduce the fluctuations (which necessarily vanish when the number of particles in a mesoscopic volume tends to infinity, since we have seen they scale as $1/\sqrt{N_s}$),

$$\delta\tilde{f}_s(\mathbf{r},\mathbf{v},t) = \tilde{f}_s(\mathbf{r},\mathbf{v},t) - f_s(\mathbf{r},\mathbf{v},t) \tag{6.13}$$

$$\delta\tilde{\mathbf{E}}(\mathbf{r}) = \tilde{\mathbf{E}}(\mathbf{r}) - \mathbf{E}(\mathbf{r}) \tag{6.14}$$

$$\delta\tilde{\mathbf{B}}(\mathbf{r}) = \tilde{\mathbf{B}}(\mathbf{r}) - \mathbf{B}(\mathbf{r}) \tag{6.15}$$

By construction, the average of the microscopic fluctuation is equal to zero. Using these definitions, we can write the ensemble averaged form of Kilmontovitch's equation as

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = -\frac{q_s}{m_s}\langle\left(\delta\tilde{\mathbf{E}}+\mathbf{v}\times\delta\tilde{\mathbf{B}}\right)\cdot\frac{\partial \delta\tilde{f}_s}{\partial \mathbf{v}}\rangle \tag{6.16}$$

The left hand of this equation describes the evolution of the smooth distribution function in phase space, under the action of the smoothed fields. This smooth distribution is the physical tool that is the most adpated for lots of plasma applications involving small scales, but mean-field, or collective, effects (i.e. not collisions, for instance). The right hand term describes the effect of the microscopic field fluctuations on the distribution function – and arise directly from the discrete, granular nature of the plasma : such a term would be equal to zero would the plasma be a smooth, continuous medium. It is the *collision integral*,

$$\mathscr{C}(f_s) = -\frac{q_s}{m_s}\langle\left(\delta\tilde{\mathbf{E}}+\mathbf{v}\times\delta\tilde{\mathbf{B}}\right)\cdot\frac{\partial \delta\tilde{f}_s}{\partial \mathbf{v}}\rangle, \tag{6.17}$$

and is in general complicated to calculate. There are several models employed to replace it, in particular diffusion terms, as we saw in the section on collisions.

In the case where this collision integral vanishes, or is negligible, the kinetic equation (6.16) is called the *Vlasov equation*. It plays a very important role in the modeling of lots of effects in space and astrophysical plasmas ; its use is justified when considering phenomena on timescales much smaller than the collision frequency of the specie s.

6.1.4 Some properties of the kinetic equation

Continuity equation in phase space

As we did for Klimontovitch's equation, we note that if the force field is divergent free in $\mathbf{v}$-space, then eq.(6.16) is a continuity equation, expressing the conservation of the particle number in phase space,

$$\frac{\partial f_s}{\partial t} + \boldsymbol{\nabla}\cdot(\mathbf{U}f_s) = \mathscr{C}(f_s), \tag{6.18}$$

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[3] under the ergodic hypothesis...

where $\mathbf{U} = (\mathbf{v}, \mathbf{F}/m_s)$ is the 6-dimensional velocity vector of the particle's flow in phase space. The collision term expresses the deviation from this conservative flow in phase space. In the absence of collision, Vlasov equation, the flow is perfectly conservative, and can be written as a total derivative

$$\frac{df_s}{dt} = 0. \tag{6.19}$$

Parametrization in phase space, drift kinetic equations

We can express the kinetic equation for any set of variables $\mathbf{x}$ one may want to parametrize the phase space with, as

$$\frac{\partial f_s}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f_s}{\partial \mathbf{x}} = \mathscr{C}(f_s). \tag{6.20}$$

A useful parametrization for particles in known slowly varying magnetic field is to introduce as variables the guiding center $\mathbf{R}_g$ of particles and the perpendicular drifts $\mathbf{V}_g$. Since in this case the magnetic moment $M = m_s v_\perp^2/2B$ is conserved, and the velocity modulus v as well, a kinetic equation (assuming no dependence on the Larmor phase ϕ) is

$$\frac{\partial f_s(v, M, \mathbf{R}_g)}{\partial t} + \mathbf{V}_g \cdot \frac{\partial f_s(v^2, \mu, \mathbf{R}_g)}{\partial \mathbf{R}_g} = \mathscr{C}(f_s). \tag{6.21}$$

in which v and μ are just conserved quantities. This kind of equation, which makes it possible to easily calculate the kinetic evolution of a population in a given magnetic field configuration, is known as a *drift-kinetic equation.*

Another interesting parametrization often found in the litterature consists in considering the gyrophased averaged motion of the particle described by the quantities $v = const.$ and $\mu = v_\parallel/v = \cos\theta$ the pitch angle cosine. Neglecting perpendicular drifts (i.e. assuming a one dimensional motion along a field line) and introducing the curvilinear coordinate along the field line s one gets

$$\frac{\partial f(v, \mu, s)}{\partial t} + \mu v \frac{\partial f(v, \mu, s)}{\partial s} + \frac{(1-\mu^2)v}{2L_B(s)} \frac{\partial f(v, \mu, s)}{\partial \mu} = \mathscr{C}(f_s). \tag{6.22}$$

where we calculated

$$\frac{d\mu}{dt} = \frac{(1-\mu^2)v}{2L_B(s)}, \qquad L_B(s) = d\ln B/ds$$

this equation, which describes the transport along a field line, is sometimes called the *focussed transport equation.*

Building steady-state solutions to the Vlasov equation

The Vlasov equation reads

$$\frac{\partial f_s}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f_s}{\partial \mathbf{x}} = 0 \tag{6.23}$$

Therefore, a distribution function that depends only on the constants of the motion C_i of a particle is always a steady-state solution of the Vlasov equation :

$$\frac{\partial f_s(C_i)}{\partial t} = 0. \tag{6.24}$$

This is a convenient way to build steady-state solutions to the Vlasov equation. For instance, a distribution function that would depend only on the particle's energy will always be a steady state solution of the Vlasov equation. You can check the classical example of the Harris current sheet in the exercises.

6.2 Link to the fluid equations

The fluid variable (n, $\mathbf{u}$, $\mathbf{p}$ etc.) are defined as the statistical moments with respect to velocity of the phase space distribution function $f(\mathbf{r}, \mathbf{v}, t)$ [4]. We define the average of ψ over the distribution function as

$$\langle \psi \rangle (\mathbf{r}, t) = \frac{1}{n(\mathbf{r}, t)} \int d^3\mathbf{v} f(\mathbf{r}, \mathbf{v}, t) \psi(\mathbf{r}, \mathbf{v}, t), \tag{6.25}$$

where the normalization factor is the number density

$$n(\mathbf{r}, t) = \int d^3\mathbf{v} f(\mathbf{r}, \mathbf{v}, t). \tag{6.26}$$

We have seen that the kinetic equation governing the evolution of the distribution function is

$$\frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}} = \mathscr{C}(f), \tag{6.27}$$

where the collision integral $\mathscr{C}$ can be attributed different functional forms, according to the model chosen, that we will not specify here - we shall only assume that there is no chemical or nuclear reaction, or particle creation/annihilation during collisions, so that the collision operator checks

$$\int d^3\mathbf{v} \mathscr{C}(f) = 0. \tag{6.28}$$

In the following, we use this evolution equation for f to derive evolution equations for the statistical moments of f – or the fluid moments of the considered specie.

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[4] In this part we forget the subscript s, but it is implicit ; there is a set of statistical moments attributed to each specie

6.2.1 Density

The number density is defined by eq.(6.26). Its evolution is obtained by integrating the kinetic equation (6.27) over velocity. We obtain

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{u} = 0 \tag{6.29}$$

where the distribution function has been assumed to vanish at infinity, and we defined the mean velocity as $\mathbf{u} = \langle \mathbf{v} \rangle$. The property of the collision operator $\int d^3\mathbf{v}\mathscr{C}(f) = 0$ has been used. This equation is usually named the *continuity equation.* It can also be expressed in term of the full derivative of the density as

$$\frac{1}{n}\frac{dn}{dt} = -\nabla \cdot \mathbf{u} \tag{6.30}$$

which illustrates the link between the divergence of the velocity field and the compressibility of the fluid. This latter expression can be used to derive a useful result : the total derivative of any scalar quantity ψ can be expressed as

$$n\frac{d\psi}{dt} \equiv n\left(\frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla\psi\right) = \frac{\partial}{\partial t} n\psi + \nabla \cdot \psi\mathbf{u}. \tag{6.31}$$

6.2.2 Momentum

We look for the evolution of the momentum density, by multiplying the kinetic equation (6.27) by $m\mathbf{v}$ and integrating over velocity. We obtain

$$\frac{\partial mn\mathbf{u}}{\partial t} + \nabla \cdot \mathbf{\Pi} = n\mathbf{F} + n\mathbf{R}. \tag{6.32}$$

The full stress tensor is defined as $\mathbf{\Pi} = mn\langle \mathbf{v}\mathbf{v} \rangle$ and the friction force $\mathbf{R}$ such as

$$n\mathbf{R} = \int d^3\mathbf{v} m\mathbf{v}\mathscr{C}(f). \tag{6.33}$$

This term describes the exchange of momentum between the population of particles described by the distribution function $f(\mathbf{r}, \mathbf{v}, t)$, and external systems through the term $\mathscr{C}(f)$. These external systems can be other populations of particles, or for instance fields fluctuations, described in a statistical manner.

The stress tensor can be separated into internal stresses and 'external' stresses, by introducing the random (or centered) velocity component $\mathbf{w} = \mathbf{v} - \mathbf{u}$. Then

$$\mathbf{\Pi} = mn\mathbf{u}\mathbf{u} + mn\langle \mathbf{w}\mathbf{w} \rangle = mn\mathbf{u}\mathbf{u} + p\mathbf{I} + \boldsymbol{\pi} \tag{6.34}$$

where the internal stress tensor $mn\langle \mathbf{w}\mathbf{w} \rangle$ has been separated between its isotropic part (the pressure tensor) and the, properly said, internal stresses $\boldsymbol{\pi}$ accounting for all the departures from isotropy.

$$\mathbf{\Pi} = mn\mathbf{uu} + \mathbf{p} \tag{6.35}$$

$$\mathbf{p} = mn\left\langle(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u}\right\rangle \tag{6.36}$$

The pressure $\mathbf{p}$ accounts for the tendency of the particle's population to expand under the action of its (centered) thermal motion.

Noting that $\nabla\cdot\mathbf{uu} = \mathbf{u}\cdot\nabla\mathbf{u} + (\nabla\cdot\mathbf{u})\mathbf{u}$, and using the continuity equation (6.29) one can re-write the equation (6.32) as

$$nm\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p - \nabla\cdot\boldsymbol{\pi} + n\mathbf{F} + n\mathbf{R} \tag{6.37}$$

$$nm\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla\cdot\mathbf{p} + n\mathbf{F} + n\mathbf{R} \tag{6.38}$$

which is the form under which is usually presented the Navier-Stokes equation.

6.2.3 Energy

We obtain the equation describing energy conservation by multiplying the kinetic equation by $\frac{1}{2}mv^2$ and integrating over velocity space. One obtains

$$\frac{\partial}{\partial t}\left(\frac{1}{2}nmu^2 + \frac{3}{2}nkT\right) + \nabla\cdot\mathbf{\Phi} = n\mathbf{u}\cdot\mathbf{F} + n\mathbf{u}\cdot\mathbf{R} + nQ \tag{6.39}$$

where the kinetic temperature is defined such as $m\mathbf{w}^2 = 3kT$. The particle energy flux density $\mathbf{\Phi}$ is defined by $\mathbf{\Phi} = n\left\langle\frac{1}{2}mv^2\mathbf{v}\right\rangle$. Q is the heat exchanged per particle and per unit time between the particle system and external macroscopic systems, it is defined by

$$nQ = \int d^3\mathbf{v}\frac{1}{2}mw^2\mathscr{C}(f), \tag{6.40}$$

and obviously

$$\int d^3\mathbf{v}\frac{1}{2}mv^2\mathscr{C}(f) = nQ + n\mathbf{u}\cdot\mathbf{R}. \tag{6.41}$$

The energy flux density can be separated into internal and convective terms,

$$\mathbf{\Phi} = n\left\langle\frac{1}{2}mw^2\mathbf{v}\right\rangle + n\left\langle\frac{1}{2}mu^2\mathbf{v}\right\rangle + n\left\langle m(\mathbf{w}\cdot\mathbf{u})\mathbf{u}\right\rangle + n\left\langle m(\mathbf{w}\cdot\mathbf{u})\mathbf{w}\right\rangle, \tag{6.42}$$

so that

$$\mathbf{\Phi} = \left(\frac{1}{2}mu^2 + \frac{3}{2}kT\right)n\mathbf{u} + (\mathbf{p}+\boldsymbol{\pi})\cdot\mathbf{u} + \mathbf{q} \tag{6.43}$$

where the heat flux density is defined as $\mathbf{q} = n\left\langle\frac{1}{2}mw^2\mathbf{w}\right\rangle$.

It is possible to obtain an equation that describes only the equation of the internal energy of the gas by subtracting the work part. This latter can be obtained by taking the scalar product of the momentum equation (6.32) by $\mathbf{u}$. Reminding that for any scalar quantity ψ one has the equality

$$n\frac{d\psi}{dt} \equiv n\left(\frac{\partial\psi}{\partial t} + \mathbf{u}\cdot\nabla\psi\right) = \frac{\partial}{\partial t}n\psi + \nabla\cdot\psi\mathbf{u}, \tag{6.44}$$

(cf formula (6.31)paragraph on the continuity equation), we can express

$$nm\mathbf{u}\cdot\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \frac{\partial}{\partial t}\frac{nmu^2}{2} + \nabla\cdot\frac{nmu^2}{2}\mathbf{u} \tag{6.45}$$

we obtain

$$\frac{\partial}{\partial t}\frac{nmu^2}{2} + \nabla\cdot\frac{nmu^2}{2}\mathbf{u} = \mathbf{u}\cdot\left(-\nabla p - \nabla\cdot\boldsymbol{\pi} + n\mathbf{F} + n\mathbf{R}\right) \tag{6.46}$$

which expresses the conservation of the macroscopic kinetic energy. Subtracting this equation from the total energy equation (6.39), we obtain

$$\frac{\partial}{\partial t}\frac{3nkT}{2} - \nabla\cdot\frac{nmu^2}{2}\mathbf{u} + \nabla\cdot\boldsymbol{\Phi} = \mathbf{u}\cdot\left(\nabla p + \nabla\cdot\boldsymbol{\pi}\right) + nQ. \tag{6.47}$$

Now using the equation (6.43) for the energy flux density, and simplifying the expression using again the equality (6.31), we finally obtain

$$n\frac{d}{dt}\frac{3kT}{2} = -\nabla\cdot\mathbf{q} - (\mathbf{p}+\boldsymbol{\pi}) : \nabla\mathbf{u} + nQ, \tag{6.48}$$

where $\mathbf{a} : \nabla\mathbf{b} = a_{ij}\partial_{x_j}b_i$, so that if $\mathbf{p} = p\mathbf{I}$, $\mathbf{p} : \nabla\mathbf{u} = p(\nabla\cdot\mathbf{u})$.

6.2.4 Entropy

The internal energy equation can be expressed, equivalently, as an equation describing the entropy production. For this we introduce the entropy per particle of a gas of non-interacting point particles in density-temperature representation,

$$S(n,T) = k\left[\ln\left(\frac{1}{n}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right) + \frac{5}{2}\right], \tag{6.49}$$

or, since the expression of the constant does not matter here,

$$S(n,T) = k\ln\frac{T^{3/2}}{n} + const. \tag{6.50}$$

Therefore the time evolution of the entropy is given by (using eq.6.48)

$$\frac{dS}{dt} + \frac{k}{n}\frac{dn}{dt} = -\frac{\nabla\cdot\mathbf{q}}{nT} - \frac{(\mathbf{p}+\boldsymbol{\pi}) : \nabla\mathbf{u}}{nT} + \frac{Q}{T} \tag{6.51}$$

then using the form $\mathbf{p} = (nkT)\mathbf{I}$, one obtains

$$\frac{dS}{dt} = -\frac{\nabla \cdot \mathbf{q}}{nT} - \frac{\boldsymbol{\pi} : \nabla \mathbf{u}}{nT} + \frac{Q}{T}. \tag{6.52}$$

So that the internal energy equation states that the increase of entropy is due to three kind of irreversible processes : heat conduction, viscosity (both internal to the gas) and exchange of heat with external systems.

6.3 The Vlasov-Poisson system in the linear approximation : kinetic plasma oscillations

We consider an ion-electron plasma, unmagnetized and perfectly collisionless. Each plasma population is described with the Vlasov equation

$$\frac{\partial f_s(\mathbf{r},\mathbf{v},t)}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s(\mathbf{r},\mathbf{v},t)}{\partial \mathbf{r}} + \frac{q_s \mathbf{E}}{m_s} \cdot \frac{\partial f_s(\mathbf{r},\mathbf{v},t)}{\partial \mathbf{v}} = 0 \tag{6.53}$$

6.3.1 The kinetic plasma parallel dielectric function

We want to find the kinetic dispersion relation of plasma waves. For this we assume the existence of a small electric field $\mathbf{E}_1(\mathbf{r}, t)$ in the plasma, associated to a small perturbations $f_{s,1}(\mathbf{r},\mathbf{v},t)$ of the distribution function, which can then be written $f_s(\mathbf{r},\mathbf{v},t) = f_{s,0}(\mathbf{v}) + f_{s,1}(\mathbf{r},\mathbf{v},t)$. We look for the expression of $f_{s,1}$ as a function of $\mathbf{E}_1$.

The linearized Vlasov equation for the specie s is

$$\frac{\partial f_{s,1}(\mathbf{r},\mathbf{v},t)}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{s,1}(\mathbf{r},\mathbf{v},t)}{\partial \mathbf{r}} + \frac{q_s}{m_s}\mathbf{E}_1 \cdot \frac{\partial f_{s,0}(v)}{\partial \mathbf{v}} = 0. \tag{6.54}$$

The Maxwell-Gauss (or Poisson) equation gives us the equation for the electric field

$$\frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{E}_1(\mathbf{r},t) = \frac{1}{\varepsilon_0}\sum_s q_s n_{s,1}(\mathbf{r},t) = \frac{1}{\varepsilon_0}\sum_s q_s \int f_{s,1}(\mathbf{r},\mathbf{v},t) d^3\mathbf{v} \tag{6.55}$$

where we assumed that the unperturbed plasma is quasi neutral, so that $\sum_s q_s n_{s,0} = 0$. We write the electric field and distribution function perturbation as continuous Fourier series (in space and time),

$$f_{s,1}(\mathbf{r},v,t) = \int\int f_{s,1k\omega}(v) e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} d^3\mathbf{k} d\omega \tag{6.56}$$

$$f_{s,1k\omega}(\mathbf{v}) = \frac{1}{(2\pi)^4}\int\int f_{s,1}(\mathbf{r},\mathbf{v},t) e^{-i\mathbf{k}\cdot\mathbf{r} + i\omega t} d^3\mathbf{r} dt \tag{6.57}$$

And for the electric field

$$\mathbf{E}_1(\mathbf{r},t) = \int\int \mathbf{E}_{1k\omega} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} d^3\mathbf{k} d\omega \tag{6.58}$$

$$\mathbf{E}_{1k\omega} = \frac{1}{(2\pi)^4}\int\int \mathbf{E}_1(\mathbf{r},\mathbf{v},t) e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t} d^3\mathbf{r} dt \tag{6.59}$$

The Vlasov equation can be written for the Fourier components as

$$i(-\omega + \mathbf{k}\cdot\mathbf{v}) f_{s,1k\omega} + \frac{q_s \mathbf{E}_{1k\omega}}{m_s}\cdot\frac{\partial f_{s,0}(v)}{\partial \mathbf{v}} = 0 \tag{6.60}$$

so that the perturbation of the distribution function is[5]

$$f_{s,1k\omega} = -\frac{i q_s \mathbf{E}_{1k\omega}}{m_s(\omega - \mathbf{k}\cdot\mathbf{v})}\cdot\frac{\partial f_{s,0}}{\partial \mathbf{v}} \tag{6.61}$$

We now express the Fourier transformed Maxwell-Gauss equation

$$i\mathbf{k}\cdot\mathbf{E}_{1k\omega} = \frac{1}{\varepsilon_0}\sum_s q_s \int f_{s,1k\omega}(\mathbf{v}) d^3\mathbf{v}, \tag{6.62}$$

and insert it in eq.(6.61). We get

$$\mathbf{k}\cdot\mathbf{E}_{1k\omega} = -\sum_s \frac{q_s^2}{m_s\varepsilon_0}\int \frac{\mathbf{E}_{1k\omega}\cdot\nabla_v f_{s,0}}{\omega - \mathbf{k}\cdot\mathbf{v}} d^3\mathbf{v} \tag{6.63}$$

This equation informs us on the component of the electric field parallel to the wavevector **k**. We introduce the notation $\|$ for the direction parallel to **k**, and $\perp$ for the component perpendicular to it. We have

$$kE^{\|}_{1k\omega}\left(1 + \sum_s \frac{q_s^2}{m_s\varepsilon_0 k}\int \frac{\nabla_{v_\|} f_{s,0}(\mathbf{v})}{\omega - k v_\|} d\mathbf{v}_\perp dv_\|\right) = 0 \tag{6.64}$$

The term in the parenthesis must be equal to zero for the Fourier components of the electric field to be non-vanishing. This term is the longitudinal dielectric function $\varepsilon^{\|}(\omega,k)$ of the plasma[6], which can be decomposed in terms of the longitudinal susceptibilities[7] of the different species $\chi^{\|}_s(\omega,k)$,

$$\varepsilon^{\|}(\omega,k) = 1 + \sum_s \frac{q_s^2}{m_s\varepsilon_0 k}\int \frac{\nabla_{v_\|} f_{s,0}(\mathbf{v})}{\omega - k v_\|} d\mathbf{v}_\perp dv_\| = 1 + \sum_s \chi^{\|}_s(\omega,k). \tag{6.65}$$

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[5] Note that we are here dividing by $(-\omega + \mathbf{k}\cdot\mathbf{v})$, which can be nul – and certainly will be, for some values of v – corresponding to resonant particles.

[6] One sees directly from eq.(6.62), adding an extra source term ρ_{ext} on the right-hand side, that the Maxwell-Gauss equation reads $ik\varepsilon^{\|}(\omega,k)E^{\|}_{1k\omega} = \rho_{ext}/\varepsilon_0$.

[7] $\chi^{\|}_s$ can also be named the polarizability of the specie s. Indeed one easily see that the parallel component of the polarization vector associated to the specie s checks $P^{\|}_{1,s,\omega k} = q_s n_{1,s,\omega k} = \varepsilon_0 \chi^{\|}_{s,\omega k} E^{\|}_{1\omega k}$

It is convenient to introduce the reduced (and normalized) distribution function $\bar{f}_{s,0}(v_\parallel)$,

$$\bar{f}_{s,0}(v_\parallel) = \frac{1}{n_0}\int f_{s,0}(\mathbf{v})d\mathbf{v}_\perp, \tag{6.66}$$

so that $\int \bar{f}_{s,0}(v_\parallel)dv_\parallel = 1$. The longitudinal susceptibilities now read

$$\chi_s^\parallel(\omega, k) = \frac{\omega_{p,s}^2}{k}\int \frac{\nabla_{v_\parallel}\bar{f}_{s,0}(v_\parallel)}{\omega - kv_\parallel}dv_\parallel \tag{6.67}$$

6.3.2 Electron plasma longitudinal waves

The dispersion relation $\varepsilon^\parallel(\omega, k) = 0$ has a regularity problem at $v_\parallel = \omega/k$, which is the wave-particle resonance : for this velocity, the electrons have exactly the phase velocity of the (ω, k) electrostatic mode, and, in the linear limit, interact "forever" with this mode, exchanging an infinite amount of energy with it. In the following, we have to assume that, $\omega - kv_\parallel$ never vanishes – for this we consider values of the phase speed $v_\varphi = \omega/k$ much larger than the thermal spread of the distribution function, so that there are practically no particles having velocities $v_\parallel = v_\varphi$; the distribution function may be assumed to be equal to zero at this point. From now on, we consider an electron plasma with ions at rest, so $\chi_i^\parallel(\omega, k) = 0$. We also drop the "parallel" indices for the parallel distribution functions.

First, we get rid of the partial derivative in the integrand in the expression of the electron susceptibility by integrating by parts

$$\int \frac{\partial_v \bar{f}_{e,0}(v)}{\omega - kv}dv = \left[\frac{\bar{f}_{e,0}(v)}{\omega - kv}\right]_{-\infty}^{\infty} - k\int \frac{\bar{f}_{e,0}(v)}{(\omega - kv)^2}dv \tag{6.68}$$

The term between bracket is nul since the distribution function vanishes at infinity. We can therefore re-express the susceptibility as

$$\chi_e(\omega, k) = -\omega_{p,e}^2\int \frac{\bar{f}_{e,0}(v)}{(\omega - kv)^2}dv. \tag{6.69}$$

Cold plasma

Neglecting the thermal motion of the electrons ($T_e \to 0$), we can approximate $\bar{f}_{e,0}(v) \simeq \delta(v)$. We then get from eqs.(6.65)-(6.69)

$$1 - \frac{\omega_p^2}{\omega^2} = 0 \quad \Rightarrow \quad \omega = \pm\omega_p \tag{6.70}$$

This is the dispersion relation that we already obtained previously in the course by neglecting the pressure term in the fluid equation.

Warm plasma

In the fluid treatment of the plasma oscillation made in chapter 2, it was not clear how to deal with the pressure term (we needed a closure relation, the choice of which was not easy to justify). The kinetic treatment overcomes this problem, and justifies the fluid closure that should be used. We assume that $\omega/k \gg v$ and Taylor expand the integrand :

$$(\omega - kv)^{-2} = \omega^{-2}(1 - kv/\omega)^{-2} = \omega^{-2}\left(1 + 2kv/\omega + 3(kv/\omega)^2 + ...\right) \tag{6.71}$$

Introducing this into eqs.(6.65)-(6.69), we get

$$1 = \frac{\omega_p^2}{\omega^2}\int \bar{f}_{e,0}(v)\left(1 + 2kv/\omega + 3(kv/\omega)^2 + ...\right) dv \tag{6.72}$$

or, introducing the mean values of the electron velocities and the wave's phase velocity $v_\varphi = \omega/k$,

$$1 = \frac{\omega_p^2}{\omega^2}\left(1 + 2\frac{\langle v\rangle}{v_\varphi} + 3\frac{\langle v^2\rangle}{v_\varphi^2} + ...\right). \tag{6.73}$$

We must assume that there is no net drift of the electron population with respect to the ion one, so $\langle v\rangle = 0$ (if not our treatment should include the presence of a net current). We get finally the dispersion relation, in the limit $\omega \simeq \omega_p$ (consistent with our assumption that $v_\varphi^2 \gg \langle v^2\rangle$) [8],

$$\omega^2 \simeq \omega_p^2 + 3k^2\langle v^2\rangle \tag{6.74}$$

Taking into account the thermal spread of the distribution function then induces a modification of the dispersion relation. The energy now propagates at a non-zero group velocity,

$$v_g = \frac{\partial\omega}{\partial k} \simeq \frac{3\langle v^2\rangle}{v_\varphi} \tag{6.75}$$

which is a small quantity with respect to the electron thermal speed, since the phase velocity had to be assumed much larger.

6.3.3 Landau damping and beam instability

We have neglected the effect of the pole (vanishing of the denominator $\omega - kv$) by considering waves fast enough for no particle to resonate with. Actually the integral appearing in the expression of the polarizability is improper around $v = v_\varphi$. Dealing with this problem requires a complicated mathematical treatment, involving integration around carefully chosen contours in the complex plane, that was made by Landau in his

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[8] Instead of this approximation, one can solve exactly the polynome to get $\omega^2 = \omega_p^2/2(1 + \sqrt{1 + 12\langle v^2\rangle k^2/\omega_p^2})$

paper *The vibration of the electronic plasma*, in 1946. The result of this analysis can be summarized though the so-called Plemelj formula,

$$\lim_{\epsilon\to 0^+}\int dx\frac{g(x)}{x-u\pm i\epsilon}\equiv \mp i\pi\int dxg(x)\delta(x-u)+\mathscr{P}\int\left(\frac{1}{x-u}\right) \tag{6.76}$$

with $\mathscr{P}$ the Cauchy principal value operator. Applying this to eq.(6.69), we have

$$\chi_s^{\parallel}=\frac{\omega_p^2}{k}\left(\mathscr{P}\int\frac{\partial_v\bar{f}_{e,0}(v)}{\omega-kv}dv-i\pi\int\partial_v\bar{f}_{e,0}(v)\delta(\omega-kv)dv\right) \tag{6.77}$$

The principal part is basically the value of the integral by setting the number of particles around the singularity equal to zero. So in the limit of large phase velocity compared to the thermal velocity of the particles, we may Taylor expand it and get the same result as in the previous section. The dispersion relation is (where we used $\delta(ax)=\delta(x)/|a|$).

$$1=\frac{\omega_p^2}{\omega^2}\left(1+3\frac{\langle v^2\rangle}{v_\varphi^2}+...\right)+i\pi\text{sign}(k)\frac{\omega_p^2}{k^2}\left.\frac{\partial\bar{f}_{e,0}(v)}{\partial v}\right|_{v=\omega/k} \tag{6.78}$$

Therefore wave frequency ω now has an imaginary part, reflecting exponential growth/damping of the mode (ω,k) with time. Separating the real and imaginary parts as $\omega=\omega_r+i\gamma$, one has

$$\omega_r(k)\simeq\omega_p\left(1+\frac{3}{2}\frac{\langle v^2\rangle k^2}{\omega_p^2}\right) \tag{6.79}$$

and

$$\gamma(k)\simeq\text{sign}(k)\frac{\pi}{2}\frac{\omega_p^3}{k^2}\left.\frac{\partial\bar{f}_{e,0}(v)}{\partial v}\right|_{v=\omega/k} \tag{6.80}$$

where it has been assumed that $\gamma\ll\omega_p$.

This result is important : electrostatic waves in plasmas constantly exchange energy with the resonant population $v=\omega/k$. This exchange takes the form of a damping (the *Landau damping*) when the slope of the distribution function around the velocity is negative. A Maxwellian plasma will then always damp waves. The rate of this damping can be very small (the damping time very long...) especially for high phase velocity waves. This is the reason why plasma waves are indeed observed in plasmas. But the damping can become very important for waves having smaller phase velocities – in particular of the order of the thermal speed of the electrons, although the treatment provided previously is in this case not exactly correct. Waves with $k\sim\omega_p/v_{th}$ will be heavily damped, and transfer their energy as thermal energy into the plasma. They can be used as heating devices for instance in Tokamaks (RF heating systems).

If the distribution function exhibit a positive slope in some region of v space (an *electron beam*), then $\gamma(k)$ will be positive in some range of wavevectors : plasma waves will grow in this region of k space. This is what is called the *beam-plasma instability.* It explains most of the high amplitude plasma waves observed in the interplanetary medium – and the consequent radio emissions associated with the propagation of electron beams in the interplanetary medium.

6.3.4 Two streams instability

This instability arises when two beams of charged particles (each beam having a thermal spread much smaller that the relative velocity between the beams) interact. We consider the normalized distribution

$$\bar{f}_0(v) = \frac{1}{2}\left(\delta(v - v_0) + \delta(v + v_0)\right). \tag{6.81}$$

The polarisability is in this case

$$\chi_e^{\|}(\omega, k) = -\frac{\omega_{p,e}^2}{2}\left(\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}\right). \tag{6.82}$$

so that the dispersion relation to solve for is

$$1 = \frac{\omega_{p,e}^2}{2}\left(\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}\right). \tag{6.83}$$

which is a bi-quadratic equation. It admits imaginary roots if its discriminent is negative, the condition for which is $v_0 > \omega_p/k$: waves will develop with phase speeds smaller than half the beams relative speed. These waves will pump kinetic energy from the beams.

6.4 Examples and exercises

6.4.1 The Harris current sheet

Current sheets are important in plasmas physics, as topological boudaries between magnetic sectors. They are subjected to various kind of instabilities, that lead to magnetic reconnection. To study this processes on the kinetic level, it is important to have a steady-state description of a current sheet in the Vlasov description. Such a model was proposed by Harris in 1962, and is still used to initialize most kinetic numerical simulations.

We consider a Cartesian frame, a current sheet thickness in the x direction and a system invariant by translation along y and z. We take the current density $\mathbf{j}$ such as $\mathbf{j} = j(x)\mathbf{u}_y$, so that $\mathbf{B} = B(x)\mathbf{u}_z$, and the potential vector is $\mathbf{A} = A(x)\mathbf{u}_y$.

To build a stationary solution, we use the property of the Vlasov equation that any distribution function of the dynamic invariants of the particles is a steady state solution. The invariance of the system along y and z implies that the particle's momentum

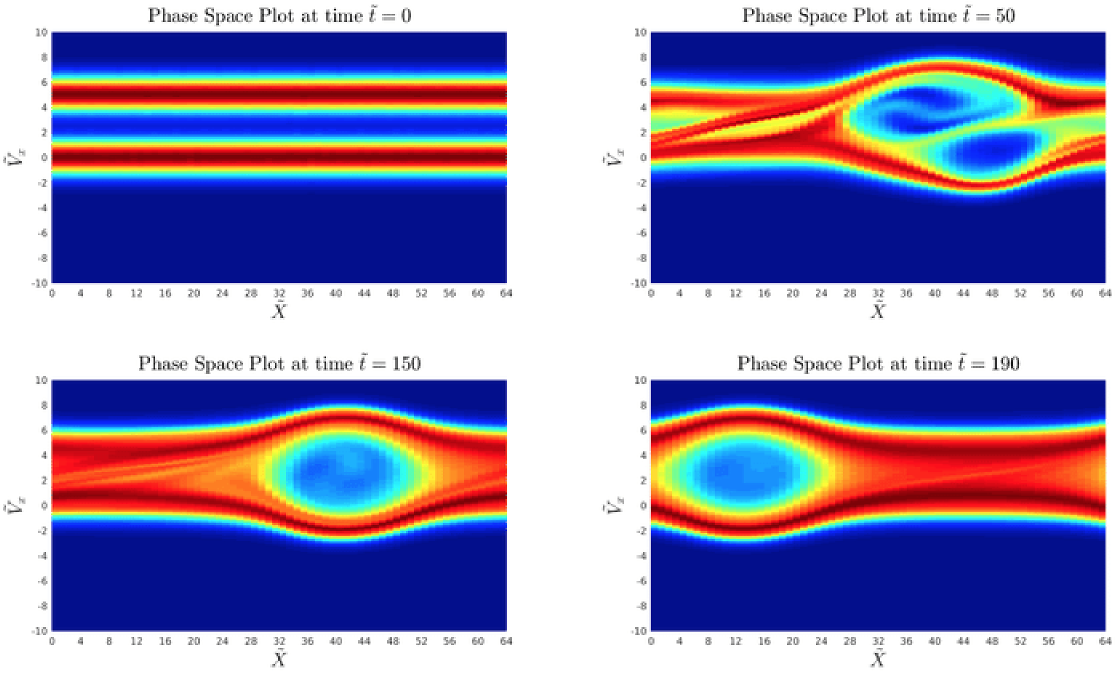

FIGURE 6.1 – Nonlinear evolution of the two-stream instability.

components p_y and p_z are constants. The energy of a particle is also a constant. Our constants read

$$E = \frac{1}{2} m_s (v_x^2 + v_y^2 + v_z^2) + q_s \Phi(x) \tag{6.84}$$

$$p_y = m_s v_y + q_s A_y(x) \equiv m_s \alpha_{2,s} \tag{6.85}$$

$$p_z = m_s v_z \equiv m_s \alpha_{3,s} \tag{6.86}$$

where we introduce the constants α homogeneous to velocities to stick to Harris paper. We now build α_1 from the energy as

$$\alpha_{1,s}^2 = 2E/m_s - \alpha_2^2 - \alpha_3^2 = v_x^2 - \frac{2q_s}{m_s} v_y A_y - \frac{q_s^2}{m_s^2} A_y^2 + 2\frac{q_s}{m_s}\Phi \tag{6.87}$$

in which we eliminated v_y^2 from the equation by using

$$v_y^2 = p_y^2/m_s^2 - q_s^2 A_y^2/m_s^2 - 2(q_s/m_s) v_y A_y. \tag{6.88}$$

Then any function $f(\alpha_1, \alpha_2, \alpha_3)$ is a steady-state solution of the Vlasov equation. Harris suggests to look for one in the form

$$f_s(x, \mathbf{v}) = \frac{n_0}{(2\pi v_{th,s})^{3/2}} \exp\left(-\frac{\alpha_{1,s}^2 + (\alpha_{2,s} - u_s)^2 + \alpha_{3,s}^2}{2v_{th,s}^2}\right) \tag{6.89}$$

with $v_{th,s}^3 = kT/m_s$. With a bit of algebra, we check that it can be recast as

$$f_s(x,\mathbf{v}) = \frac{n_0}{(2\pi v_{th,s})^{3/2}} \exp\left(-\frac{q_s(u_s A_y + \Phi)}{kT}\right) \exp\left(-\frac{v_x^2 + (v_y - u_s)^2 + v_z^2}{2v_{th,s}^2}\right) \tag{6.90}$$

We now need to couple this equation to the field. For this we first compute the density and current by integrating on velocity space, we get

$$n_s(x,\mathbf{v}) = n_0 \exp\left(-\frac{q_s(u_s A_y + \Phi)}{kT}\right) \tag{6.91}$$

and

$$\mathbf{j}_s(x,\mathbf{v}) = n_0 u_s \exp\left(-\frac{q_s(u_s A_y + \Phi)}{kT}\right) \mathbf{u}_y = n_s u_s \mathbf{u}_y \tag{6.92}$$

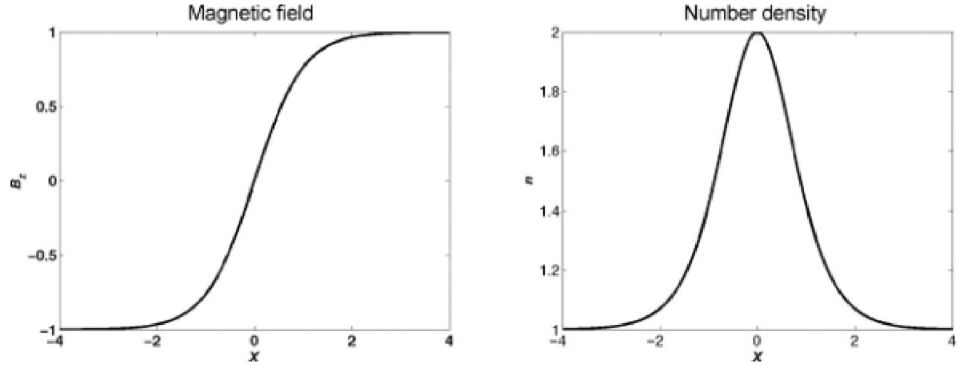

Figure 6.2 – Magnetic field and density profile in a Harris current sheet.

The Poisson equation reads

$$\frac{d^2\Phi}{dx^2} = \frac{en_0}{\varepsilon_0}\left(\exp\left(\frac{e(u_e A_y + \Phi)}{kT}\right) - \exp\left(\frac{-e(u_i A_y + \Phi)}{kT}\right)\right) \tag{6.93}$$

we simplify the problem by assuming that $u_e = -u_i = U$ (which is always possible by a proper change of frame), we get

$$\frac{d^2\Phi}{dx^2} = \frac{2en_0}{\varepsilon_0} \exp\left(\frac{eUA_y}{kT}\right) \sinh\frac{e\Phi}{kT} \tag{6.94}$$

The vector potential also satisfies a Poisson equation [9],

$$\frac{d^2 A_y}{d^2 x} = -\mu_0 j_y, \tag{6.95}$$

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[9] From Maxwell-Ampère, using rot(rot) formula and noting that the divergence of A is equal zero from the chosen form $\mathbf{A} = A(x)\mathbf{u}_y$.

from which we get

$$\frac{d^2A_y}{d^2x} = -2\mu_0 e n_0 U \exp\left(\frac{eUA_y}{kT}\right)\cosh\frac{e\Phi}{kT} \tag{6.96}$$

Eq.(6.94) has the trivial solution $\Phi = 0$, which is the only solution for which the plasma is strictly speaking quasi-neutral : we retain this one. We still have to solve eq.(6.96). Multiplying by A'_y on both side and integrating we get

$$\frac{1}{2}A'^2_y = C - 2\mu_0 n_0 kT \exp\left(\frac{eUA_y}{kT}\right) \tag{6.97}$$

The solution to eq.(6.96) is obtained after some complicated calculations

$$A_y(x) = -\frac{2kT}{eU}\ln\cosh\left(\frac{x}{\lambda_D}\frac{U}{c}\right) \tag{6.98}$$

where $\lambda_D^2 = \varepsilon_0 kT/ne^2$ as usual. The density of both species is

$$n(x) = n_0 \exp\left(\frac{eUA_y}{kT}\right) = \frac{n_0}{\cosh^2\left(\frac{x}{\lambda_D}\frac{U}{c}\right)}. \tag{6.99}$$