Exam O3 Space plasmas

Feb. 9th, 2021

Exercice 1 : Parker's model and the interplanetary electric field

Our purpose is to establish a steady state ($\partial_t \equiv 0$) solar wind model, in which the plasma is considered to be unmagnetized ($\overrightarrow{\mathrm{B}} = \overrightarrow{0}$), and consists of an electron fluid (e indices) and a proton fluid (p indices), coupled to each other by an electric field. These two populations are described by the 3 first moments of the distribution functions ($n_e, \overrightarrow{u_e}, \mathrm{T}_e$) and ($n_p, \overrightarrow{u_p}, \mathrm{T}_p$). We use the following notations : $\overrightarrow{\mathrm{E}}$ is the electric field, M the Sun's mass, G the gravitational constant, k Boltzmann's constant, e proton's charge, m_p proton's mass, m_e the electron mass.

The system exhibits a spherical symmetry, so that all physical quantities are function of r only (r being the distance to the center of the Sun). One notes : $\overrightarrow{\mathrm{E}} = \mathrm{E}(r)\vec{e}_r$, $\overrightarrow{u_e} = u_e(r)\vec{e}_r$, $\overrightarrow{u_p} = u_p(r)\vec{e}_r$.

The electron and proton fluids are assumed to be isothermal (T_e and T_p do not depend on r).

1) Write the equations that express the conservation of density and conservation of momentum in the electron and proton fluids.

In the following, the plasma is assumed to be quasi-neutral, and the current is assumed to be equal to zero everywhere.
a) Give a justification for these two assumptions.
b) What do these assumptions imply for the physical quantities n_e, u_e, n_p, u_p ?

3) Show that in the limit of massless electrons ($m_e \to 0$), the electric field can be expressed as

$$\mathrm{E} \simeq -\frac{k\mathrm{T}_e}{en_e}\frac{dn_e}{dr}$$

4) Show that this relation between E and the electron pressure gradient is equivalent to postulate a Boltzmann equilibrium for the electron population in the electrostatic potential $\varphi(r)$ associated to $\overrightarrow{\mathrm{E}}$.

5) In the following, the protons are supposed to be much cooler than the electrons ($\mathrm{T}_p \ll \mathrm{T}_e$). Show that the electric field in the plasma can be expressed as

$$\mathrm{E} \simeq \frac{m_p u_p}{e}\frac{du_p}{dr} + \frac{\mathrm{GM}m_p}{er^2}.$$

Quickly discuss the physical meaning of these two terms.

6) Show that the proton velocity field $u_p(r)$ is a solution of a differential equation having the form

$$\frac{(u_p/c_s)^2 - 1}{u_p}\frac{du_p}{dr} = f(r)$$

give the expression of the function $f(r)$, in which a characteristic radius r_c will be introduced. What is the expression and physical meaning of r_c ? What is the expression and physical meaning of c_s ?

7) Plot, on a (r, u_p) diagram, the different families of solution of $u_p(r)$, and quickly comment on the physcial nature of each of them.

8) Let $u_{ts}(r)$ be the solar wind's velocity field in the *trans-sonic* case. Integrating the equation obtained in question 6, give a relation linking $u_{ts}(r)$ and r.

9) Show that when $r \gg r_c$, the speed $u_{ts} \sim 2c_s\sqrt{\ln r/r_c}$. Show that this implies that, far away from the Sun, the interplanetary electric field decreases as $\mathrm{E} \sim \mathrm{U}_0/r$. Give the expression for the potential U_0.

10) Do you expect this estimation of $\mathrm{E}(r)$ to be valid everywhere in the interplanetary medium ? If not, give (very approximately) the region in which you expect this expression to be valid.

Exercice 2 : Ionospheric cut-off frequency

In this problem we try to evaluate an order of magnitude of the cutoff frequency of radio waves in the terrestrial ionosphere. We consider the ionisation by solar UV of the upper atmosphere. Let q_{em} be the maximum of the ionisation rate per unit volume. Chapman's model gives an expression :

$$q_{em} = \frac{\mathrm{N}_\nu(\infty)\cos\chi}{\exp(1)\mathrm{H}}.$$

1) Give the physical meaning of the different quantities appearing in this expression.

2) Using the numerical parameters provided below, give an order of magnitude of q_{em} in the Earth's ionosphere. (Show the calculations through which you obtain your result).

3) We consider that ions and electrons recombine with a dissociative reaction

$$\mathrm{XY}^+ + e^- \rightarrow \mathrm{X}^* + \mathrm{Y}^*$$

for which the reaction constant is k_{dis}. Give an order of magnitude of n_e, the maximum concentration of electrons in the Earth's ionosphere. (Show the calculations through which you obtain your result)

4) Obtain from the previous estimates the order of magnitude of the cutoff frequency of radio waves in the Earth's ionosphere.

Numerical parameters :

- Mass of molecular nitrogen $\mathrm{M}_{\mathrm{N}_2} \simeq 47 \times 10^{-27}$ kg
- Electron mass $m_e \simeq 9,1 \times 10^{-31}$ kg
- $\mathrm{N}_\nu(\infty) \simeq 5 \times 10^{14}\ \mathrm{m}^{-2}.\mathrm{s}^{-1}$
- $k_{dis} \simeq 7 \times 10^{-14}\ \mathrm{m}^3.\mathrm{s}^{-1}$
- Boltzmann's constant $k \simeq 1,4 \times 10^{-23}$ J.K^{-1}.
- Electron charge $e \simeq 1,6 \times 10^{-19}$ C
- Vacuum permittivity $\epsilon_0 \simeq 8,9 \times 10^{-12}$ F.m^{-1}

Appendix : Fluid equations

— Continuity equation (conservation of the number of particles), which relates the density n (scalar) to the average velocity $\mathbf{u}$ (vector) :

$$\frac{\partial n}{\partial t} + \text{div } n\mathbf{u} = 0 \tag{1}$$

— Conservation of momentum equation, which relates the average velocity $\mathbf{u}$ (vector) to the pressure p (in the general case a tensor of order 2, we will consider in this course, to avoid complicating things too much, isotropic media, and thus a scalar pressure $p = nk\mathrm{T}$).

$$mn\left(\frac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + n\mathbf{F} \tag{2}$$

— The conservation of energy equation, which relates the internal energy (or pressure) to the heat flow $\mathbf{j}_{th}$,

$$\frac{\partial}{\partial t}\left(nm\frac{u^2}{2} + \frac{3}{2}p\right) + \text{div}\left[\mathbf{u}\left(nm\frac{u^2}{2} + \frac{5}{2}p\right) + \mathbf{j}_{th}\right] = n\mathbf{u}\cdot\mathbf{F} + \mathrm{Q} \tag{3}$$

In this equation, the term $\mathbf{j}_{th}$ describes heat transport by thermal conduction, while Q (in [W.m^{-3}]) describes heat transfer from a source external to the population of particles under consideration ($\mathrm{Q} > 0$ if heat enters the system).