Introduction to Space Plasma physics

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Chapter 1: Ionization and recombination

From planetary environments to the intergalactic medium, most of the visible matter in the universe is ionized. The main reason for this is the existence of stars, which are sources of ionizing radiations.

The plasma state results from the balance between two competing processes : ionization, which is the production of a positive ion and a free electron from a neutral atom, and recombination, which is the inverse process. The two major ways to ionize an atom are by electron impact, or by photo-ionization. In this chapter, we briefly describe how to quantify these effects.

1.1 Thermal equilibrium

In thermal equilibrium, the ionization degree of a gas resulting from the balance between all ionizing processes (collisions, radiation...) is given by Saha's equation.

$$\frac{n_e n_{X^{n+1}}}{n_{X^n}} = \frac{2g_{X^{n+1}}}{g_{X^n}} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-W/kT} \tag{1.1}$$

n_e is the free electron density, X^n and X^{n+1} are two consecutive ionization ground states of the atomic specie X. W is the potential of ionization, i.e. the difference of energy between the ground states X^n and X^{n+1}.

The result of this equation in the case of a gas of hydrogen, for different densities of nuclei $n_{tot} = n_{H^+} + n_H$ is illustrated on Fig.1.1. We see that the ionization degree is high even for temperature quite smaller than the ionization energy. This is because, at low densities, recombination is a rather inefficient process.

The mass density of the photosphere of the Sun (which is an opaque medium in which Saha's law is applicable) is around 3×10^{-4} kg.m^{-3}, which corresponds to a number density of protons of $n = 2 \times 10^{23}$ m^{-3}. The observed temperature is $T = 6400$ K. Saha's equation gives a value of $\alpha \simeq 4 \times 10^{-4}$: the photosphere is rather neutral.

In the above layers, Saha's equation is not valid because the plasma is optically thin, and not in a thermal equilibrium state. This happens to be the case in most dilute plasmas, which makes Saha's equation of little concrete use in plasma physics.

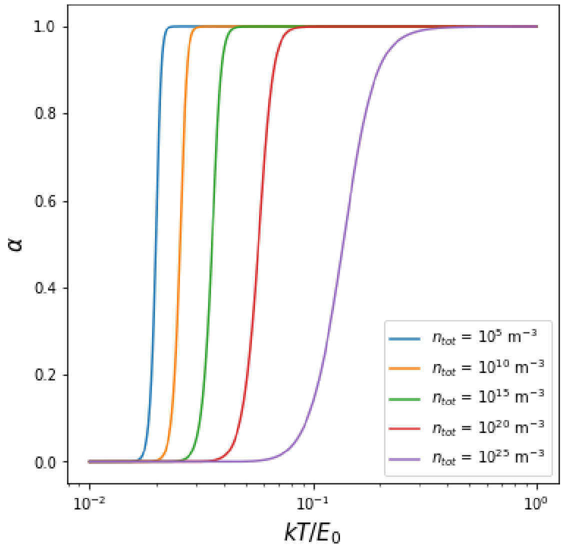


FIGURE 1.1 – Ionisation degree $\alpha = n_e/(n_e+n_{H^+})$ as a function of the temperature for different densities.

1.2 Out of equilibrium description

The out of equilibrium description of a plasma passes by the description of each possible ionization and recombination reaction, by the measure of its efficiency, and by the calculation of the ionization profiles in a particular geometrical configuration. We show these steps in the following.

1.2.1 Reaction cross-section

The efficiency of a reaction is characterized by its cross-section. For instance, consider

$$A + B \rightarrow C + D. \tag{1.2}$$

Consider a volume containing "target" particles B at rest. A flux density $\Phi_A = n_A v_{A/B}$ of particles A is passing through the volume (n_A and v_A are the number density and velocity of the particles A). The number of reaction per unit volume and unit time is

$$\dot{n}_{reac} = dn_C/dt = -dn_A/dt = n_A v_{A/B} n_B \sigma \tag{1.3}$$

which defines the cross section σ, homogeneous to a surface. Of course the equation is unchanged by exchanging indices A and B ($v_{A/B} = v_{B/A}$ is here a positive value).

We can also define the mean-free path λ of the specie A "colliding" with the background made of specie B from the small probability dp that A interacts with B while travelling a small distance ds : $\lambda^{-1} = dp/ds$. Then we have

$$n_A(s + ds) = n_A(s)(1 - dp) \Rightarrow dn_A/ds = -n_A/\lambda \quad (1.4)$$

from the definition of λ. So, if a beam of particles A is launched in a medium characterized by a constant mean-free path, its density with decrease exponentially along its way, with a characteristic decay length equal to λ.

Now making the connection with the cross section : we simply have the distance travelled $ds = v_{A/B}dt$, so the number of reaction per unit time and volume is, from the previous equation

$$dn_A/dt = -n_A v_{A/B}/\lambda \Rightarrow \lambda = (n_B\sigma)^{-1} \quad (1.5)$$

1.2.2 Ionization processes

The two main ionization reactions are the ionization by electron impact and the photo-ionization. The first correspond to the equation

$$A + e^- \to A^+ + e^- + e^- \quad (1.6)$$

and is characterized by a cross section σ_e given, in the classical approximation, by the Thomson formula [1]

$$\sigma_e(E) = \frac{\pi q_e^4(E - W)}{WE^2} \quad (1.7)$$

where E is the ionizing electron's energy in the rest frame of the impacted atom, and $q_e \equiv e/\sqrt{4\pi\varepsilon_0}$. The maximum of this function is reached at twice the ionisation potential W, and is $\sigma_e \sim 10^{-20}$ m^2 for $W \sim 10$ eV.

The photoionisation reaction involves the interaction with a photon,

$$A + h\nu \to A^+ + e^-. \quad (1.8)$$

It is characterized by a cross section σ_{ph} that is a function of the energy $h\nu$ of the photon and of the chemical properties of A. There is no simple, classical expression for this cross section, but a good order of magnitude, in practical cases, is given by $\sigma_{ph} \simeq 5 \times 10^{-22}(E/W)^{-3}$ m^2, for photo-electrons having energies larger than the ionisation potential.

The ionising source in most astrophysical cases will be a star. In this case the flux density of ionising photons at a distance r from the star will be related to the star spectral luminosity L_ν by

$$N_{ph} = \frac{1}{4\pi r^2}\int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} d\nu \quad (1.9)$$

1. A demonstration of this formula will be given in the chapter on collisions.

where $\nu_0 = W/h$ is the minimal ionisation frequency, and, approximating the star radiation by a perfect blackbody,

$$L_\nu(R,T) = 4\pi R^2 \frac{2\pi h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \tag{1.10}$$

where R and T are the star radius and temperature, respectively. Typical numbers, for the Sun, $4\pi r^2 N_{ph} \sim 1.8 \times 10^{33}$ photons.s^{-1}. For a B type main sequence star ($T = 30000$ K), $4\pi r^2 N_{ph} \sim 1.5 \times 10^{48}$ photons.s^{-1}.

1.2.3 Recombination processes

Recombination is the inverse process of ionisation. The ionisation reactions seen above can be read in the opposite direction, giving the three-body recombination

$$A^+ + e^- + e^- \to A + e^- \tag{1.11}$$

which is in most dilute plasma cases a rather inefficient process.

The inverse of the photoionisation is the radiative recombination,

$$A^+ + e^- \to A + h\nu \tag{1.12}$$

which is an important process in astrophysics. It is usually, in the absence of molecular species, the dominent process, and is characterized by a cross section $\sigma_{rad} \sim 10^{-24}$m^2. Note that it is much smaller than the ionisation cross section.

In the presence of diatomic molecules, there exist a much more efficient recombination process, which is the dissociative recombination

$$AB^+ + e^- \to A^* + B^* \tag{1.13}$$

Its cross section is $\sigma_{dis} \sim 10^{-18}$ m^{-2} : this process happens to be dominent in most regions of the Earth's ionosphere, which is mainly constituted of N_2 or O_2 molecules.

1.3 A few example, or exercices

1.3.1 Chapman's ionization layer

An example of a space plasma (mainly) produced by photo-ionization is the Earth's ionosphere. Here we develop a model for the structure of the ionospheric plasma layer.

We assume that the atmosphere is composed by a single chemical specie of molecular mass m, in hydrostatic equilibrium at temperature T with a base density n_0, so that the molecular number density is

$$n(z) = n_0 e^{-z/H}, \qquad H = \frac{kT}{mg} \tag{1.14}$$

g is the acceleration of gravity, assumed constant. We assume this exponential atmosphere is impacted by solar ionizing radiation coming along the direction of the z axis, and characterized by a photon flux density $N_{ph}(z)$ in $\mathrm{m}^{-2}\mathrm{s}^{-1}$.

The probability that a photon gets absorbed when it travels a distance dz is $dp = dz/\lambda_{ph} = n(z)\sigma_{ph}dz$. Therefore the flux of photons evolves with z according to

$$N_{ph}(z-dz) = N_{ph}(z)(1-dp) \Rightarrow \frac{dN_{ph}}{dz} = n(z)\sigma_{ph}N_{ph} \tag{1.15}$$

from which we obtain

$$N_{ph}(z) = N_{ph}(\infty)\exp\left(-\sigma_{ph}\int_z^\infty n(z)dz\right) \tag{1.16}$$

where we have assumed $n(z)$ is a given function (i.e. the ionization process does produce an ion/electron density that is small compared to the neutral density). The argument of the exponential is called the optical depth of the atmosphere at altitude z. $N_{ph}(\infty)$ is the ionizing photon flux from the Sun at the Earth's orbit.

We can calculate $N_{ph}(z)$ using eq.(1.14) :

$$N_{ph}(z) = N_{ph}(\infty)\exp\left(-\sigma_{ph}n(z)H\right) \tag{1.17}$$

which gives the profile of the Sun's UV light in the atmosphere. According to eq.(1.8), an electron is created each time a photon is absorbed. So, the number electron created per unit volume checks $Q_e dz = N_{ph}(z+dz) - N_{ph}(z)$, and $Q_e = dN_{ph}/dz$. We can evaluate this quantity using the previous results,

$$Q_e = -n(z)\sigma_{ph}N_{ph}(z) = N_{ph}(\infty)\sigma_{ph}n_0\exp\left(-\sigma_{ph}n(z)H - z/H\right) \tag{1.18}$$

that is often put into the form

$$Q_e = Q_{max}\exp\left(1 - y - \exp -y\right) \tag{1.19}$$

and is called the Chapman's electron production function, from the British plasma physicist Sydney Chapman.

The number density of electrons in the ionosphere is now obtained by assuming steady-state equilibrium between electron production and recombination. The dissociative recombination is dominent, and occurs with a volumic rate R, according to eq(1.13),

$$R = k_{dis}n_e^2, \qquad k_{dis} = \sigma_{dis}v_e \tag{1.20}$$

v_e being the thermal speed of the free electrons. The steady-state $R = Q_e$ gives the electron number density in the ionosphere

$$n_e(z) = \sqrt{Q_e(z)/k_{dis}} \tag{1.21}$$

FIGURE 1.2 – The Rosette Nebula, a paradigmatic example of a Strömgren sphere.

1.3.2 Strömgren sphere

For this exercise, only order of magnitude estimations and qualitative reasonings are asked for.

Assume that we have a 30.10^3 K B-type main sequence star surrounded by an interstellar medium composed entirely of hydrogen, with density $n_H = 1$ cm^{-3}. Consider a spot at 5 pc from the star.

1. How long will a neutral hydrogen atom stay neutral ?
2. What is the fraction $\xi = n_{H^0}/n_H$ of neutral hydrogen (assuming the only recombination process is radiative) ?
3. What is the radius R of the ionized region ? To answer this question, you will introduce the radiative recombination coefficient α^* for all recombinations into excited levels of H. Indeed, radiative recombinations to the ground state produce again an ionizing photon and must not be counted for the ionization equilibrium. The recombination coefficient is $\alpha^* \simeq 2.6 \times 10^{19} (T/10^4\ \text{K})^{-0.7}$ m^3.s^{-1}, and is defined so that the number of recombination per unit volume is $\dot{n}_{rec} = \alpha^* n_e n_{H^+}$.
4. What is the typical extent of the boudary between the ionized region and the neutral gas around ?

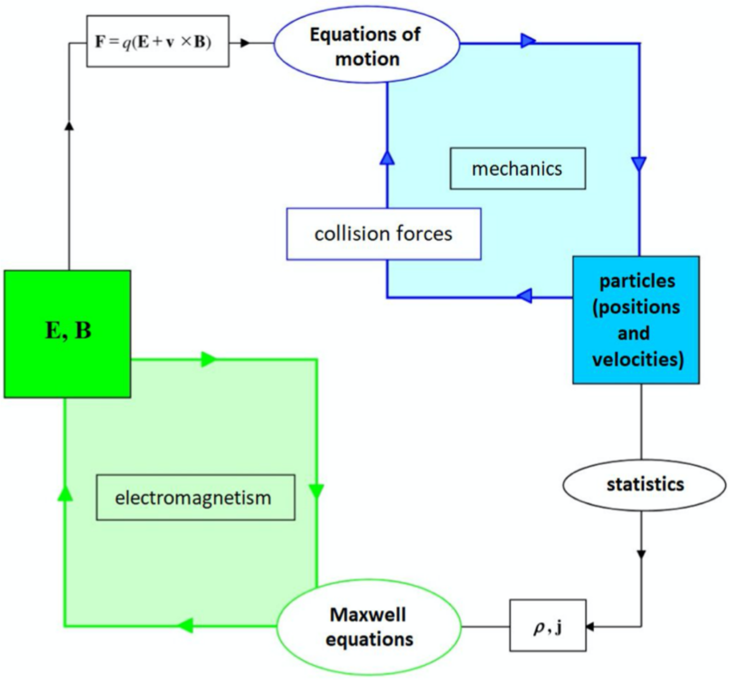
1.3.3 A cometary ionosphere

A comet emits H_2O molecules at a speed V, with a flux density F_0, in a spherical symmetry. It is illuminated by solar ionizing radiation, incoming with a photon flux N_{ph}. Calculate the ionization profile around the comet.

Indication : first calculate the radial profile $n_n(r)$ of neutral water molecules around the comet. Then write the equation for the time evolution of the density of ionized water H_2O^+. Assuming steady-state, deduce the radial profile of H_2O^+ around the comet. (You will assume that the optical depth of the cometary environment is always close to 1, so the UV flux can be assumed constant in the whole volume of the cometary atmosphere).

Chapter 2: Plasma scales, and collective phenomena

The dynamics of neutral gas are controlled by short range interactions between constituants, i.e. collisions. Plasma are different because of the effect of the Lorentz force, which is at the origin of long range, macroscopic correlations in the plasma. This effect is intrinsically non-linear, as it involves feedback effects between the field and the ionized matter.



To summarize the feedback loop : the charged particles phase-space distributions are determined by the action of long-range electromagnetic forces, which are themselves determined by the charged particles phase-space distributions through the Maxwell's equations. Plasma physics is largely related to the study of this class of coupled problems.

2.1 Plasma scales

We have a gas of charged particles of mass m and charge e, coupled through the electrostatic field characterized by the constant ϵ_0. The gas has a density n and a thermal energy kT.

A quick dimensional analysis [1] may help us identifying some typical scales of such an "electrostatic plasma". We look for length or timescales in the form $(\epsilon_0/e^2)^\alpha (m)^\beta n^\gamma (kT)^\delta$. For lengths, we have the system

$$\begin{cases} -\alpha+\beta+\delta=0 \\ -3\alpha-3\gamma+2\delta=1 \\ 2\alpha-2\delta=0 \end{cases} \Rightarrow \begin{cases} \alpha=\delta \\ \beta=0 \\ \gamma=-(\alpha+1)/3 \end{cases}$$

We have three equations for 4 unknowns, so the system is underdetermined (we can build an infinity of length scales). Interesting examples are obtained for

- $\alpha = 0$ gives $\ell = n^{-1/3}$, this is the interparticular distance of the gas.
- $\alpha = -1/2$ gives $\lambda_D = (\varepsilon_0 kT/ne^2)^{1/2}$ is the Debye length, which is the electrostatic screening length in a plasma.
- $\alpha = -1$ gives $\lambda_L = e^2/(\varepsilon_0 kT)$, this is, to some multiplicative factor, the Landau length (or the "thermal" Landau length), which si the distance at which the thermal energy between particles equals their potential interaction energy, and determines the typical distance for large-angle collisions in a plasma.

Note that these three scales are not independent, since we obviously have $\lambda_D^2\lambda_L \sim \ell^3$.

Using the same procedure, we can also find a timescale, and a typical velocity,

- $c_s = (kT/m)^{1/2}$ is the usual (isothermal) sound speed.
- $\tau_p^{-1} = \omega_p = (ne^2/m\varepsilon_0)^{1/2}$ is the plasma frequency, which is the typical oscillation frequency of a charge density perturbation in the plasma.

If we consider that the plasma is magnetized, other characteristic times and scales appear. The plasma is now characterized by the value of its "macroscopic" magnetic field B [2], and we must add to the analysis the constant μ_0 (or equivalently, the light velocity c).

1. $[\epsilon_0/e^2] = M^{-1}L^{-3}T^2$, $[n] = L^{-3}$ and $[kT] = ML^2T^{-2}$
2. We did not consider a macroscopic electric field since, as we shall see, such a field cannot really exist in the plasma

We can now build the time and length scales associated to individual particle's dynamics : $\omega_c = qB/m$ and $\rho = c_s/\omega_c$, which are the gyrofrequency, and the thermal Larmor radius.

But supplementary "collective" scales appear, involving the plasma density. From the expression of the energy density $\mu_0 B^2/2$, we can obtain a velocity $V_A = (B^2/nm\mu_0)^{1/2}$, and a new associated length $\lambda_i = V_A/\omega_c$. They are called the Alfvén speed, and the inertial length (sometimes the London length, in the context of magnetic screening).

In the following, we will study in more details the physical meaning of these scales, and see in what context they appear.

2.2 Electroneutrality : the steady-state limit

As a medium rich in free-electrons, a plasma is of course a good electric conductor. And as a good conductor, it cannot properly hold non-zero charge density in its volume, at least not for a long time (compared to $\tau = \varepsilon_0/\sigma$, where σ is the electrical conductivity). This can be seen from a simple reasoning : in a macroscopic homogeneous steady state (i.e. $\partial_t = 0$ and $\partial_{\mathbf{r}} = 0$), the electric field must nearly vanish [3] : $mdv/dt = 0 \sim qE$. Then, nearly everywhere in the plasma, one has $\rho = \varepsilon_0 \text{div}\, \mathbf{E} \simeq 0$.

A steady-state plasma is then expected to neither contain any macroscopic electric field, or charge density. If we introduce a charged object in a plasma, the electrons and ions will configure themselves so that they screen this charge. We see how in the next paragraph.

2.2.1 Electrostatic screening

The Debye length is associated to the collective phenomena of electrostatic screening in a plasma : a charged object will attract charged particles of the opposite sign, and repulse charges of the same sign, creating a non-neutral sheath around the object. The charge density in the sheath will screen the object's charge and ensure quasi-neutrality in the rest of the plasma volume.

Assume an infinite conducting plane, and the perpendicular axis z. The plane carries a uniform surface charge σ. On $z > 0$ we have a plasma of ions of mass M and charge e and electrons of mass m. Far away from the plane, we assume the plasma is unperturbed, and overall neutral. We assume steady state, the density of the ions and electrons is given by

$$n_e(r) = n_\infty \exp(e\varphi(z)/kT), \qquad n_i(r) = n_\infty \exp(-e\varphi(z)/kT) \tag{2.1}$$

3. the nearly comes from the fact that the electric field may compensate for other fields of force, we discuss this in the "ambipolar electric field section"

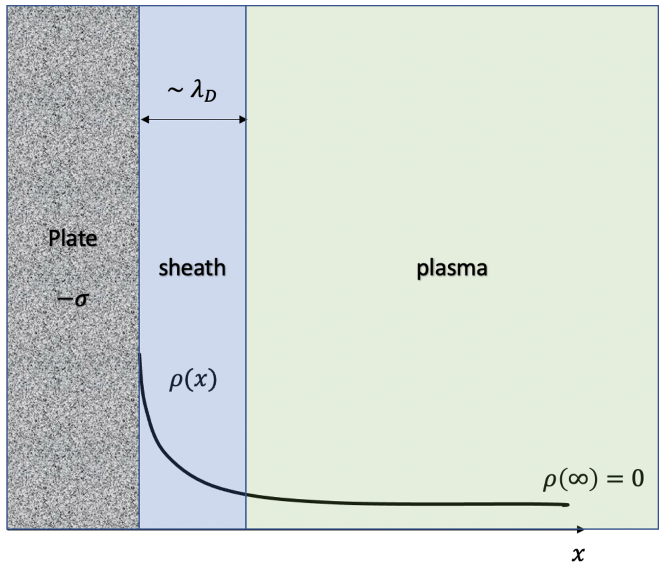


FIGURE 2.1 – Sheath structure around a macroscopic object embedded in a plasma.

where $\varphi(\infty) = 0$ has been assumed. The Poisson equation gives the potentiel $\varphi(z)$ in the plasma,

$$\Delta\varphi(r) = -\frac{2en_\infty}{\varepsilon_0}\sinh\left(\frac{e\varphi(r)}{kT}\right). \tag{2.2}$$

We now assume that the thermal energy of the particles is substantially larger than the electric potential, so that $\sinh e\varphi/kT \simeq e\varphi/kT$. Poisson equation now reads

$$\frac{d^2}{dz^2}\varphi + \frac{2n_\infty e^2}{\varepsilon_0 kT}\varphi = 0, \tag{2.3}$$

The solution of which is

$$\varphi = \varphi_0 e^{-z/\lambda_D}, \qquad \lambda_D^2 = \frac{\varepsilon_0 kT}{2n_\infty e^2} \tag{2.4}$$

We see that the potential of the charged plate is screened over a Debye length. The electric field at $z = 0$ must be equal to $\sigma_{plate}/\varepsilon_0$ (from the Gauss theorem). So we can determine the value of φ_0 as

$$E(0) = \lambda_D^{-1}\varphi_0 = \sigma_{plate}/\varepsilon_0 \Rightarrow \varphi_0 = \frac{\sigma_{plate}\lambda_D}{\varepsilon_0} \tag{2.5}$$

We now look at the charge density in the Debye sheath :

$$\rho(z) = e(n_i - n_e) = \frac{-2e^2 n_\infty}{kT}\varphi(z) = -\frac{\varepsilon_0\varphi(z)}{\lambda_D^2} \tag{2.6}$$

So, the charge per unit surface of the sheath is

$$\sigma_{sheath} = \int \rho(z)dz = -\frac{\varepsilon_0 \varphi_0}{\lambda_D} = -\sigma_{plate}. \tag{2.7}$$

The charge in the sheath is opposite to the one carried by the plate and the whole system, once integrated, is neutral.

2.2.2 The plasma parameter, and scale ordering

Consider a screened point charge q placed in a plasma. The potential around the charge is

$$\varphi(r) \sim \frac{q}{4\pi\varepsilon_0 r} e^{-r/\lambda_D}. \tag{2.8}$$

The calculation of the Debye length has assumed that the plasma particles kinetic energy was much larger than their electrostatic potential energy, $e\varphi/kT \ll 1$. From eq.(2.8), we see that this correspond to the condition [4]

$$\Gamma = \frac{1}{n\lambda_D^3} \ll 1 \tag{2.9}$$

which is equivalent to saying that there needs to be a large number of plasma particles in the Debye sphere for our assumption to be valid, and for screening to be possible.

Γ is called the plasma parameter. It is defined as the inverse of the number of particles in a Debye sphere ; it needs to be very small compared to one for a gas of charged particles to exhibit screening, and collective behaviours associated to it, and hence to be called a "plasma" properly speaking. Astrophysical plasma typically all show $\Gamma \ll 1$. For instance in the solar wind (which is a very dilute plasma), at 1 AU, one has $\lambda_D \simeq 10$ m and $n \simeq 5$ cm^{-3}, from which $\Gamma \simeq 2 \times 10^{-10}$.

The smallness of Γ makes it possible to order the plasma scales ℓ, λ_L and λ_D introduced in the beginning of this chapter. We have $\ell/\lambda_D = \Gamma^{1/3} \ll 1$, and $\ell/\lambda_L = (\lambda_D/\ell)^2 = \Gamma^{-2/3} \gg 1$, so plasma scales have the following important ordering

$$\lambda_L \ll \ell \ll \lambda_D, \tag{2.10}$$

with ratios controlled by the value of the plasma parameter Γ.

2.2.3 The ambipolar electric field

An interesting effect arises from quasi-neutrality, in the presence of density, or pressure, gradients of the charged species. Consider isothermal electrons and ions at temperature T, placed in an acceleration field $\mathbf{g}(\mathbf{r})$. In the absence of an electric field, the

4. we evaluate an electron potential energy at $r \sim \lambda_d$ from the point charge.

densities of electrons and ions, which have very different masses, should be very different, leading to the existence of space chrage ρ... So we have to assume that an electric field exists.

We can calculate it by assuming that it forces local quasi-neutrality everywhere in the plasma : $n_i(\mathbf{r}) = n_e(\mathbf{r}) = n(\mathbf{r})$. In the static limit, the gradient is determined by the force equilibrium

$$0 = -kT\nabla n + nq_\alpha \mathbf{E}(\mathbf{r}) + nm_\alpha \mathbf{g}(\mathbf{r}) \tag{2.11}$$

where $\alpha = e, i$. By substracting the two equations, we obtain the electric field in the plasma [5]

$$\mathbf{E}(\mathbf{r}) \simeq -\frac{m_i \mathbf{g}(\mathbf{r})}{2e}. \tag{2.12}$$

The "effective mass" of an ion in this electric field is now $m_i' = m_i/2$, and the effective mass of an electron is $m_e' = m_i/2$, as can be seen by replacing the value of the electric field in eq.(2.11).

This type of electric field, arising from the neutrality of the plasma on large scales, is called an ambipolar electric field. An classic example of such a field is the Pannekoek-Rossland electric field, that exists in plasma atmospheres (in particular, stellar atmospheres) and roughly doubles the scale-height of a plasma atmosphere with respect to a neutral one.

2.3 Deviation from quasi-neutrality : the plasma time response

We considered in the previous section plasmas in steady-state, that are effectively quasi-neutral. This limit must be valid as long as we consider timescales long enough for the plasma particles to maintain the space-charge sheath around charged objects, or for frequency quite smaller than

$$\omega_p = v_e/\lambda_D = \sqrt{\frac{n_0 e^2}{m_e \epsilon_0}}. \tag{2.13}$$

Here we see appearing the timescale that was derived in the beginning of this section, and that we call the plasma frequency (or plasma "angular frequency" to be rigorous, but angular is most of the time omitted). We will see that enforced deviations from quasi-neutrality oscillate at this frequency.

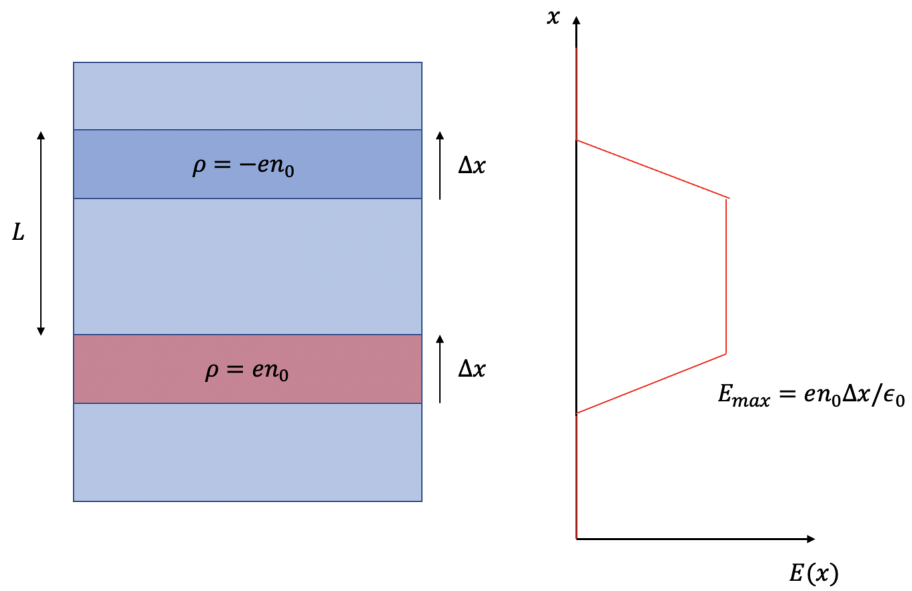


FIGURE 2.2 – Plasma oscillation : heuristic treatment.

2.3.1 The plasma oscillation

A heuristic approach

We consider a quasi-neutral plasma of electrons and ions, having each the density n_0 and charge $\pm e$. We displace a layer of width L of electrons by a small displacement $\Delta x \ll L$, creating two regions of non-zero charge density (cf. fig.2.2). The system is assumed invariant by translation along the two directions perpendicular to the x axis, so all quantities only depend on the x coordinate. We look for the time evolution of this initial configuration, and assume that we can consider the ions to be motionless (we will check at the end that it is the case).

The electric field in the plasma is given by the Maxwell-Gauss equation

$$\frac{dE}{dx} = -\frac{e(n_0 - n_e(x))}{\varepsilon_0} \Rightarrow E(x) \sim en_0\Delta x/\epsilon_0 \tag{2.14}$$

in most of the region between 0 and L, since we assume that Δx is a very small quantity with respect to L. Outside of the region $0 < x < L$, the electric field is zero : the plasma is quasi-neutral with a nul electric field. Inside this region, the electrons move according to

$$\frac{d^2\Delta x}{dt^2} = -\frac{eE}{m} = -\frac{e^2 n_0}{m\varepsilon_0}\Delta x, \tag{2.15}$$

5. we neglect the electron mass compared to the ion mass

so that the electrons oscillate at the plasma frequency $\omega_p = (n_0 e^2/m\varepsilon_0)^{1/2}$. The electric field in the plasma is

$$E(0 < x < L, t) \simeq \frac{e n_0 \Delta x_0}{\epsilon_0} \cos \omega_p t \tag{2.16}$$

Under the action of this electric field, the amplitude of the ion motion is

$$\Delta x_{ions} \sim \frac{e^2 n_0}{m_i \epsilon_0 \omega_p^2} \Delta x_0 = \frac{\omega_{p,ions}^2}{\omega_p^2} \Delta x_0 = \frac{m_e}{m_i} \Delta x_0 \tag{2.17}$$

which is smaller than the electron one by a factor at least $\sim$ 2000. This justifies the motionless ions hypothesis.

The linearized fluid equations approach

We now consider the time evolution of a small electron density perturbation using the system of fluid equations. The plasma is assumed homogeneous and quasi-neutral, with density n_0, and the ions are assumed to stay at rest, as in the previous part.

As in the previous paragraph, the initial density perturbation is only along the x coordinate of a cartesian frame, so that the problem is invariant by translations along the y and z axis.

The evolution of the electron fluid is given by the following equations (the fluid moments are $n_e(x,t)$: electron density, $u_e(x,t)$: electron mean velocity, $p_e(x,t)$: electron pressure)

$$\partial_t n_e + \partial_x (n_e u_e) = 0 \tag{2.18}$$

and

$$n_e m_e \left(\partial_t u_e + u_e \partial_x u_e\right) = -\partial_x p_e - e n_e E. \tag{2.19}$$

The electric field is coupled to the electron density through the Maxwell-Gauss equation,

$$\partial_x E = -\frac{e(n_e - n_0)}{\varepsilon_0} \tag{2.20}$$

Eqs.(2.18)-(2.19)-2.20) form a typical example of a non-linear set of coupled equations between particles and field dynamics typical of plasma collective behaviours, as discussed in the introduction of this chapter.

In order to proceed, we first note that we have 3 equations but 4 unknowns (n_e, u_e, p_e and E). So we miss an equation. This is a problem typical of the fluid treatments, for which a "closure" expressing the higher order moment (here the pressure) as a function of the smaller order ones is always required [6]. Here we shall use the "cold plasma" closure,

6. the question of how to find such physically justified closure is an important research problem in plasma physics, of particular importance when collisions are rare

$p_e = 0$.

In order to proceed, we shall assume that the perturbation is small, and linearize these equations around some equilibrium state. For this we assume that $n_e(x,t) = n_0 + \delta n_e(x,t)$, with $\delta n_e \ll n_0$ everywhere, $u_e(x,t) = 0 + \delta u_e(x,t)$ and $E(x,t) = 0 + \delta E(x,t)$.

The linearized equation set is

$$\partial_t \delta n_e + n_0 \partial_x \delta u_e = 0 \tag{2.21}$$

$$n_0 m_e \partial_t \delta u_e = -e n_0 \delta E. \tag{2.22}$$

$$\partial_x \delta E = -\frac{e \delta n_e}{\varepsilon_0} \tag{2.23}$$

Deriving the first equation with respect to time and using the two others to express δE and δu_e, we obtain

$$\frac{\partial^2}{\partial t^2} \delta n_e + \omega_p^2 \delta n_e = 0. \tag{2.24}$$

We see that an initial localized charge density perturbation will be maintained in time through oscillations at the plasma frequency. Of course we neglected here all effects related to viscosity and collisions, and these so-called plasma oscillations are in fact damped over time, on a timescale of the order of the inverse of the electron collision frequency in the plasma.

This frequency of the plasma is of fundamental importance, since the plasma will tend to react to any kind of electrostatic perturbation by oscillating at ω_p : radio antennas plunged in space plasmas constantly measure such oscillations (of more or less large amplitudes).

2.3.2 Propagation of an electromagnetic transverse wave in a plasma

In this part, we investigate the propagation of a transverse plane wave of frequency ω in a plasma. We follow the procedure of the previous section and linearize the equations for the electron motion, and assume the ions as motionless.

The electric field evolution is given by

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \Delta \mathbf{E} = -\mu_0 \frac{\partial \mathbf{j}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{2.25}$$

we look for plane wave solutions, $\mathbf{E} \propto \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t)$

$$\Delta \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \mathbf{j}}{\partial t}. \tag{2.26}$$

We used $\nabla \cdot \mathbf{E} = 0$, as can be demonstrated from Maxwell-Ampère equation, combined with the fact that $\mathbf{E}$ and $\mathbf{j}$ are co-linear, as shown below.

The current density $\mathbf{j}$ is obtained from the electrons dynamics in the wave's field. One has (in linearized form)

$$\mathbf{j} = n_0 e \mathbf{u}_e \tag{2.27}$$

and the electron macroscopic velocity perturbation evolution is coupled to E through eq.(2.22),

$$\frac{\partial \mathbf{u}_e}{\partial t} = -\frac{e}{m_e}\mathbf{E}, \tag{2.28}$$

so that the wave equation is now

$$\Delta \mathbf{E} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right)\mathbf{E} = 0 \tag{2.29}$$

where we used $\mu_0 = 1/c^2\varepsilon_0$.

The dispersion relation for the electromagnetic wave in the plasma is then

$$\omega^2 = \omega_p^2 + k^2c^2 \tag{2.30}$$

showing that transverse waves with frequencies smaller than the plasma frequency will be evanescent (imaginary k). It also shows that the phase speed $v_\varphi = \omega/k$ of electromagnetic waves in a plasma is larger than the speed of light [7]. This produces the interesting effect that electromagnetic radiation, when penetrating a plasma interface, will be refracted away from the normal to the surface, in opposition to intuitive behaviour for classical optical systems. This can be see from the optical refractive index of a plasma :

$$n = \frac{kc}{\omega} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} < 1. \tag{2.31}$$

This optical property is used to transmit long distance communications, by reflection on the ionosphere. The typical plasma frequency of this latter is around 10 MHz.

2.4 Examples and exercises

2.4.1 Exercise : capacitance of a conductive sphere in a plasma

We consider a sphere of radius a, within a plasma constituted by single charged ions of density n_i and electrons of density n_e. The problem is of spherical symmetry with respect to the center of the sphere, hence the densities are function of the distance r to

7. Of course one can check that the group velocity v_g is always smaller than c. You can demonstrate as an exercise that $v_\varphi v_g = c^2$

this center only.

We want to calculate the capacitance of the sphere. For this we assume a device that maintains it at a potential V with respect to "infinity" – the potential far away being the one of the unperturbed plasma, that we assume to be equal 0.

First, calculate the potential $\varphi(r)$ around the sphere. (Trick : you may introduce $\psi(r) = r\varphi(r)$ to simplify the Poisson equation in spherical coordinates, and solve for ψ).

Then calculate the charge carried by the sphere, assuming overall neutrality of the sphere + plasma system. Deduce the capacity of the conductive sphere. In what is it different from the capacity of a sphere in vacuum ?

2.4.2 Example : a plasma atmosphere, and the Pannekoek-Rossland electric field

An application of the previous calculation is the one of a plasma atmosphere. We assume a constant gravitational field $\mathbf{g} = -g\mathbf{u}_z$. In the absence of an electric field, one would have the following density profiles

$$n_\alpha(z) = n_{\alpha,0} e^{-z/H_\alpha}, \qquad H_\alpha = \frac{kT}{m_\alpha g} \tag{2.32}$$

where $\alpha = e, i$. We can see that $H_e \gg H_i$: the electron atmosphere extends to much higher altitudes than the ion atmosphere does, because of the small electron mass. Therefore the high altitude region is negatively charged and the low altitude one positively charged, leading to the existence of an electric field, and to a contradiction (since the density profiles were obtained not taking into account any electric field). The correct hydrostatic equations are

$$-kT\frac{dn_\alpha(z)}{dz} + n_\alpha q_\alpha E(z) - n_\alpha m_\alpha g = 0 \tag{2.33}$$

Making the assumption that $n_e \simeq n_i \simeq n(z)$ everywhere, and summing the equations (2.33) for ions and electrons, we obtain

$$-2kT\frac{dn}{dz} - n(m_e + m_i)g = 0 \Rightarrow n(z) = n_0 e^{-z/H}, \qquad H = \frac{2kT}{m_i g} \tag{2.34}$$

where we neglected m_e with respect to m_i. Therefore we show the scale-height of the ions is twice the one it would be in a neutral atmosphere (the ions and electrons both have now an effective mass $m_i/2$). The ambipolar electric field responsible for this effect is usually named the Pannekoek-Rossland electric field (Pannekoek and Rossland studied this effect in the solar atmosphere in the 1930's), $E_{PR}(z) = m_i g/2e$.

2.4.3 Exercise : the ambipolar diffusion

We consider the diffusive motion of a cloud of plasma in a neutral gas of density independent of $\mathbf{r}$. We consider that the collisions between the plasma particles and the neutral gas produce a friction force $\mathbf{f}_\alpha = \nu_\alpha \mathbf{u}_\alpha$. With ν_α the collision frequency between neutral and the population α. We assume that the friction dominates over convection, so that the motion of the population α is essentially diffusive and is described by the equation

$$0 = -kT\nabla n_\alpha + n_\alpha q_\alpha \mathbf{E}(\mathbf{r}) + n_\alpha \nu_\alpha \mathbf{u}_\alpha \tag{2.35}$$

What would the diffusion coefficient of each population if there was no electric field ? What would be the problem, given that the particles are not neutral ? What is the actual "ambipolar" diffusion coefficient of the plasma in the neutral gas ?

2.4.4 Exercise : Dispersion relation of a "hot plasma"

Instead of considering the cold plasma closure, suppose that the closure is isothermal $p_e = n_e kT_e$ with $T_e = const.$

1. Find the dispersion relation of the plasma waves by looking for normal modes $\sim \exp(kx - \omega t)$. What interesting difference arises when taking the temperature effect into account ?
2. Same question for a polytropic closure of the type $d_t(p_e n_e^{-\gamma}) = 0$. What relevant value of γ would you consider ?

Chapter 3: Collisions in fully ionized plasmas

In this chapter, we study the notion of Coulomb collisions, i.e. collisions of charged particles - which are controlled by the long range coulomb force (at the opposite of short range dipole-dipole interactions controlling collisions between atoms or molecules). These collisions are of central importance in the physics of fully ionized plasmas.

We will limit ourselves to the "Lorentzian plasma approximation", in which electrons are colliding on infinitely massive ions. In such a model, there cannot be energy exchange between electrons and ions [1], only momentum exchange. It is enough to describe several interesting transport properties of the plasma, while keeping the math simple.

3.1 Large angle deflections

The deflection angle α of an electron when passing by a ion depends on the impact parameter b. To have an angle $\alpha \sim 1$, we need the electron to pass close enough from the ion for the electrostatic interaction energy to be of the order of its kinetic energy [2]

$$\frac{Zq_e^2}{b} \sim \frac{1}{2}mv^2 \Rightarrow b \sim \frac{2Zq_e^2}{mv^2} \equiv b_L(v). \tag{3.1}$$

This "Landau impact parameter" b_L is defined as the electron-ion distance for which the electrostatic interaction energy equals the kinetic energy, and is related to the thermal Landau length by $b_L(v) = \lambda_L(v_{th}/v)^2$.

Therefore, we define the cross-section for large angle collisions as $\sigma_{l.a.} = \pi b_L^2(v)$. The associated electron mean-free path is $\lambda_{l.a.} = (n\sigma_{l.a.})^{-1}$ where n is the density of ions, and the mean time between collisions is $\Delta t \sim \lambda/v \sim (n\sigma_{l.a.}v)^{-1}$. The mean change in particle angle per unit time under the action of large angle collisions is $\Delta\alpha/\Delta t \sim n\sigma_{l.a.}v \propto v^{-3}$. One must note the very strong decrease in the efficiency of the Coulomb collisions with the electron's speed (we will recover the same effect for small angle collisions). This dependency has critical consequences for plasma physics, that we will discuss in the following.

1. For instance, ions and electrons populations of different temperatures cannot relax to a thermalized system

2. we use in whole of this section the notation $q_e \equiv e/\sqrt{4\pi\varepsilon_0}$

We have seen in the previous chapter (sec. 2.2.2) that in a plasma $\lambda_L \sim \Gamma^{2/3}\ell \ll \ell$, so the Landau length will usually be by order of magnitude larger than the interparticular distance : we can then expect large angle collisions to be very unfrequent. An order of magnitude of their mean-free path is

$$\lambda_{l.a.} \sim (n\sigma_{l.a.})^{-1} \sim \frac{\ell^3}{\lambda_L^2} \sim \Gamma^{-2}\lambda_L \sim \Gamma^{-1}\lambda_D. \tag{3.2}$$

In the following, we will calculate the mean-free path associated to small angle collisions, and see that large angle deviations are indeed negligible in a plasma.

3.2 Small angle deflections

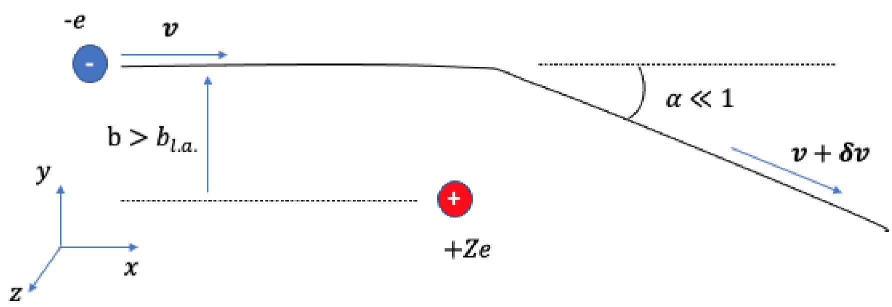


FIGURE 3.1 – Geometry of the small angle collision considered

We consider collisions with impact parameter b much larger than $b_L(v)$ calculated in the previous paragraph. These collisions produce only a very small deflection $\delta\mathbf{v}$ in the velocity of the electron, so, we may calculate this deflection by assuming that the trajectory $\mathbf{r}(r)$ of the electron is a straight line

$$\mathbf{r}(t) = \mathbf{b} + \mathbf{v}t \tag{3.3}$$

with $\mathbf{b}$ the impact parameter (cf. fig.3.1). $t = 0$ is the time when the closest approach is reached, and the ion is supposed at rest at the origin of the coordinate system. The distance between the electron and the ion is

$$r(t) = \sqrt{b^2 + v^2t^2}. \tag{3.4}$$

The electric force experienced by the electron at position $\mathbf{r}(t)$ is

$$\mathbf{F}(t) = -\frac{Zq_e^2\mathbf{r}(t)}{r^3(t)}. \tag{3.5}$$

The total variation of its velocity vector during its travel by the ion is

$$\delta\mathbf{v} = \int_{-\infty}^{\infty} \frac{d\mathbf{v}}{dt} dt = \frac{-1}{m_e} \int_{-\infty}^{\infty} \frac{Zq_e^2 \mathbf{r}(t)}{r^3(t)} dt \tag{3.6}$$

We can evaluate separately the parallel and perpendicular (with respect to the electron initial velocity vector) component of the small velocity variation as

$$\delta v_x = \frac{-1}{m_e} \int_{-\infty}^{\infty} \frac{Zq_e^2 vt}{r^3(t)} dt = 0 \equiv \delta v_\parallel \tag{3.7}$$

since the integrand is odd : obviously, the effect of the electric field on this direction when the particle arrives to the ion ($t < 0$) cancels the one when it leaves the ion ($t > 0$) and there is no net change in the parallel component of the electron [3]. This is different for the perpendicular component [4],

$$\delta v_y = \frac{-1}{m_e} \int_{-\infty}^{\infty} \frac{Zq_e^2 b}{r^3(t)} dt = \frac{-2Zq_e^2}{bm_e v} = -\frac{b_L(v)}{b} v \equiv \delta v_\perp \tag{3.8}$$

The deflection angle α of the electron during this interaction is

$$\alpha(b) \simeq \frac{|\delta v_y|}{v} = \frac{b_L(v)}{b}. \tag{3.9}$$

The Landau radius $b(v)$ determines the typical distance above which our assumption of small angle deflection is valid. Indeed for impact parameters much larger than b_L, $\alpha \ll 1$ and approximating the electron motion by a straight line is justified. For impact parameters much smaller, the deflection is by large angles and our treatment is not correct.

Consider a typical collision with an impact parameter equal to the interparticular length ℓ and a velocity equal to the thermal velocity. The deflection angle given by eq.(3.9) is $\alpha = \lambda_L/\ell \sim \Gamma^{2/3} \ll 1$. This shows again that typical collisions in plasmas are by very small angles (reminding that, for instance in the solar wind, $\Gamma \sim 10^{-10}$, typical collisions make the direction of the velocity of the electron vary by $\sim 10^{-6}$ rad).

3.2.1 Angular scattering

Now that we have investigated the case of the interaction between two particles, we calculate the time evolution of the deflection angle of the electron under the action of its interaction with all the ions around. The number of targets encountered on a length element $d\ell$ along the electron's trajectory is

$$dN = n 2\pi b db d\ell \tag{3.10}$$

3. This is because we assumed a straight line trajectory, and because the ion is assumed at rest. Including these effects will provide a non-zero $\delta v_\parallel$, but that will always be of much smaller magnitude than the change $\delta v_\perp$

4. You may demonstrate that a primitive of $(a + bt^2)^{-3/2}$ is $(t/a) \times (a + bt^2)^{-1/2} + const.$

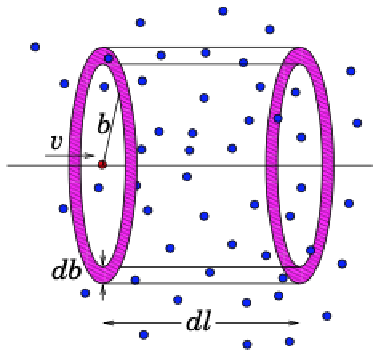


FIGURE 3.2 – Volume containing the encountered targets along $d\ell$.

as can be seen on fig.3.2. The mean deflection $\langle \Delta \mathbf{v}_\perp \rangle$ of the electron during the time Δt is obtained by summing the deflections $\delta \mathbf{v}_\perp$ due to all these interactions. We first reformulate eq.(3.8) in a vector form as

$$\delta \mathbf{v}_\perp = -\frac{b_L(v)v}{b}\frac{\mathbf{b}}{b} \tag{3.11}$$

and obtain

$$\frac{\langle \Delta \mathbf{v}_\perp \rangle}{\Delta t} = n b_L(v) v^2 \int_0^{2\pi} \mathbf{u}_b(\phi) d\phi \int db = 0 \tag{3.12}$$

where $\mathbf{u}_b(\phi) = \mathbf{b}/b$ is the unit radial vector in cylindrical coordinates. We see that the deflection compensate themselves in average : the direction of the electron velocity vector is not changed.

But we can check that the evolution of the variance $\langle \Delta v_\perp^2 \rangle$, increases with time. Indeed the mean square deflection undergone by the electron during the time Δt is

$$\frac{\langle \Delta v_\perp^2 \rangle}{\Delta t} = 2\pi n b_L^2(v) v^3 \int \frac{db}{b}. \tag{3.13}$$

Unfortunately, this integral, taken between 0 and ∞, diverges both in 0 and in ∞... The divergence for $b \to \infty$ comes from the fact that the coulomb interaction is long range ($1/r^2$). But in a plasma, Debye screening shortens the effective range of the Coulomb force. So, the integral can be regularized "on physical bases" by limiting the integration at high values of b to λ_D – therefore assuming that the interaction potential becomes nul after this distance. The integral also diverges for $b \to 0$. This is because the deflection angles become infinite for zero value of the impact parameter. This (incorrect) result comes from our treatment which assumed small angle deflections. So, we cut the integral for low values of b at $b_L(v)$ – therefore assuming that there are no ions in a range of distance to the electron smaller than λ_L. We finally obtain for the variation,

$$\frac{\langle \Delta v_\perp^2 \rangle}{\Delta t} = 2\pi n b_L^2(v) v^3 \ln \Lambda \tag{3.14}$$

We introduced the Coulomb logarithm $\ln\Lambda = \ln\lambda_D/\lambda_L \sim \ln\Gamma^{-1}$. In most plasma of interests (whether natural or artificial), the thermal value of the Coulomb logarithm is $\ln\Lambda \sim 15 - 25$.

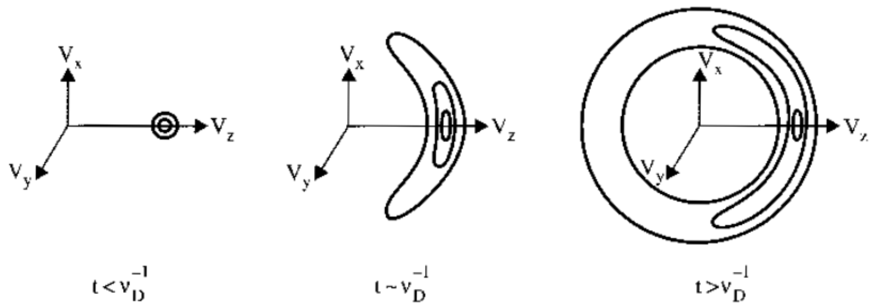


FIGURE 3.3 – Angular diffusion with a characteristic scattering time ν_D^{-1}, from Helander and Sigmar 2005 citing Trubnikov 1965.

We see that the variance of the deflection angle $\alpha = v_\perp/v$ increases linearly with time. This is characteristic of an angular diffusion process : the electron undergoes lots of small angle random deflections. The mean effect is not a net average change in the velocity direction, but an increase of spread of the probability to find the electron in a given direction. The diffusion coefficient characterizing this process is

$$D_\alpha = \frac{\langle \Delta\alpha^2 \rangle}{2\Delta t} = \pi n b_L^2(v) v \ln\Lambda = \nu_{ei}\left(\frac{v_{th}}{v}\right)^3 \tag{3.15}$$

where we introduced the thermal electron-ion collision frequency

$$\nu_{ei} = \pi n \lambda_L^2 v_{th} \ln\Lambda = \frac{4\pi n Z^2 q_e^4 \ln\Lambda}{m_e^2 v_{th}^3} \tag{3.16}$$

The collision frequency is then defined, for small angle Coulomb collisions, as the time it takes for the diffusion effect to substantially spread the distribution of probability to find an electron at a given angle with respect to a given axis.

To conclude this part, note that the collision frequency obtained $\nu_{ei}(v)$ is equal to the one that we obtained for large angle collisions in the first section, multiplied by the Coulomb logarithm $\ln\Lambda$. It gives another interpretation of the Coulomb logarithm (and of the plasma parameter Γ) as the ratio of the efficiency of small to large angle collisions to scatter the plasma particles. Since $\ln\Lambda \sim 20$, we see (once again) that small angle scattering is completely dominant in a fully ionized plasma.

3.2.2 Dynamical friction force and related effects

The angular scattering that has just been described is necessarily accompanied by a slowing down (a friction) of the electron in the parallel direction. This comes from the fact that the total energy $\mathscr{E}$ of the electron is conserved in the scattering process. One has

$$\Delta\mathscr{E} = \frac{1}{2}m_e\left((v+\Delta v_\parallel)^2 + \Delta v_\perp^2 - v^2\right) = 0 \tag{3.17}$$

from which we have,

$$2v\Delta v_\parallel + \Delta v_\parallel^2 + \Delta v_\perp^2 = 0. \tag{3.18}$$

Since $\Delta v_\parallel \ll v$, the first term is much larger than the second one, which can be neglected. Taking the average of the remaining terms, we get the expression of the parallel slowing down

$$\frac{\langle\Delta v_\parallel\rangle}{\Delta t} = -\frac{1}{v}\frac{\langle\Delta v_\perp^2\rangle}{2\Delta t} = -\nu_{ei}\frac{v_{th}^3}{v^2} \tag{3.19}$$

Since this parallel component is applied along the electron velocity vector $\mathbf{v}$, the friction force on the electron due to the angular scattering can be expressed as

$$\mathbf{f} = \frac{m_e\langle\Delta v_\parallel\rangle}{\Delta t}\frac{\mathbf{v}}{v} = -m_e\nu_{ei}\frac{v_{th}^3}{v^3}\mathbf{v} \tag{3.20}$$

This is an important and general result : the existence of angular scattering with a diffusion coefficient D_α produces a dynamical friction force on the scattered particles $\mathbf{f} = -m_e D_\alpha \mathbf{v}$. We now investigate a few interesting effects directly linked to the existence of this friction force.

Runaway electrons and the Dreicer electric field

An important collisional process in plasma physics is called the runaway effect. It is due to the fact that the collision frequency decreases strongly with the velocity of the electron ($\propto v^{-3}$). Therefore, an electron in an electric field may gain in average energy from the field in spite of the collisions. The proper description of this effect necessitate a kinetic treatment ; we propose here a simplified depiction, providing correct orders of magnitude.

Consider an electron in a constant electric field $\mathbf{E}$, colliding with background ions. Its kinetic energy (actually its mean "directed" kinetic energy), checks

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -e\mathbf{E}\cdot\mathbf{v} - m_e\nu_{ei}\frac{v_{th}^3}{v}. \tag{3.21}$$

from which we see that an electron having a velocity v_{lim} such that

$$v_{lim}(E) = \left(\frac{m_e\nu_{ei}v_{th}^3}{eE}\right)^{1/2} \tag{3.22}$$

will neither lose nor gain energy from the field (it is somehow in equilibrium). An electron with a velocity $v < v_{lim}$ will be "overdamped", which means that its velocity will in average decrease under the action of the collisional friction. A particle with a velocity $v > v_{lim}$ will be "underdamped", which means that its energy will increase in time without limit under the action of the electric field. Such electrons are called "runaway" electrons. Note that, as small as the applied electric field can be, there will always be a small fraction of runaway electrons in the tail of the velocity distribution function.

The Dreicer electric field is the value of the electric field for which the thermal electrons become themselves runaway – at this stage no stable plasma can really exist. Its value is given by $v_{lim}(E_D) = v_{th}$,

$$E_D = \frac{m_e \nu_{ei} v_{th}}{e}. \tag{3.23}$$

We can now conveniently express $v_{lim}(E) = (E_D/E)^{1/2} v_{th}$.

Runaway electrons are important in plasma physics, since they can take a lot of energy out of the electric field (they can be responsible for the breaking of plasma state in lab experiments). They also appear to play an important (although not clearly elucidated) role in lots of astrophysical plasmas.

Subsonic fluid friction and electrical conductivity

We want to calculate the friction force $\mathbf{f}$ acting, not on a single particle, but on a small electron fluid volume of density n_e and mean velocity $\mathbf{u}$. This is

$$\mathbf{f} = -m_e n_e \langle \nu_{ei} \frac{v_{th}^3}{v^3} \mathbf{v} \rangle = -m_e n_e \nu_{ei} v_{th}^3 \int \frac{\mathbf{v} d^3\mathbf{v}}{v^3} f(\mathbf{v}) \tag{3.24}$$

where $f(\mathbf{v})$ is the distribution of the electron fluid velocities, so that the probability to find an electron with a speed between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ is $dp = f(\mathbf{v}) d^3\mathbf{v}$. We assume that our fluid element is near-equilibrium, so that $f(\mathbf{v})$ is a Gaussian with thermal speed v_{th} and a drift speed $\mathbf{u}$ such that $u \ll v_{th}$. We have

$$f(\mathbf{v}) = \frac{1}{(2\pi)^{3/2} v_{th}^3} e^{-(\mathbf{v}-\mathbf{u})^2/2v_{th}^2} \simeq \frac{1}{(2\pi)^{3/2} v_{th}^3} e^{-v^2/2v_{th}^2} \left(1 + \frac{\mathbf{v} \cdot \mathbf{u}}{v_{th}^2}\right) \tag{3.25}$$

The integral on the first term is equal to 0, since the integrand is odd. Let's define z along the vector $\mathbf{u}$. The x and y components of $\mathbf{f}$ are equal to 0 (odd integrands). So the only component left is along z, and is equal to

$$f_z = -\frac{m_e n_e \nu_{ei} u}{(2\pi)^{3/2}} \int \frac{v_z^2 d^3 v}{v_{th}^2 v^3} e^{-v^2/2v_{th}^2} \tag{3.26}$$

The integral is calculated in spherical coordinates (with the convenient change of variable $\mu = \cos\theta$, so $d^3v = v^2 dv d\mu d\phi$),

$$\int_0^{2\pi} d\phi \int_{-1}^{1} \mu^2 d\mu \int_0^{\infty} \frac{v dv}{v_{th}^2} e^{-v^2/2v_{th}^2} = 2\pi \times \frac{2}{3} \times 1 = \frac{4\pi}{3}. \tag{3.27}$$

We finally obtain

$$\mathbf{f} = -m_e n_e \frac{2\nu_{ei}}{3\sqrt{2\pi}}\mathbf{u} \equiv -\frac{m_e n_e \mathbf{u}}{\tau_{ei}}. \tag{3.28}$$

So, there is a factor $2/3\sqrt{2\pi}$ between the viscous force on the fluid and the force on a single electron that would have a velocity equal to the thermal velocity. In several references, the friction timescale $\tau_{ei} = 3\sqrt{2\pi}/2\nu_{ei}$ is introduced, as in the previous equation.

The expression of this viscous force makes it possible to easily calculate the electric conductivity σ of a plasma. We consider the ions at rest, so that the current density and the electric field are linked by $\mathbf{j} = -en\mathbf{u} = \sigma\mathbf{E}$. We consider values of the electric field much smaller than E_D, so that we can neglect runaway effects : all the conduction electrons are assumed to be underdamped, so that a steady state can be reached in the plasma. The equation describing this steady state is

$$0 = -e\mathbf{E} - m_e\mathbf{u}/\tau_{ei}, \tag{3.29}$$

so that the steady-state value of the fluid velocity is

$$\mathbf{u} = -\frac{e\tau_{ei}}{m_e}\mathbf{E} \tag{3.30}$$

from which we find the value of the plasma conductivity

$$\sigma = \frac{ne^2\tau_{ei}}{m_e} \tag{3.31}$$

Space plasmas are, to rare exceptions, never really close to equilibrium (the collisional mean-free paths are usually, for a large part of the electron energy distributions, much larger than the typical gradient scales of the system considered). The expressions of the conductivity given here is thus to be taken very cautiously.

3.3 Examples and exercises

3.3.1 Ionization cross section

Although a bit outside of the Coulomb collision topic, it is interesting to note that we can use a procedure similar to the one previously employed in order to derive the Thomson formula, seen in the first chapter, for the classical ionization cross section of an atom by electron impact.

We consider the energy transfer from an electron moving in a straight line with impact parameter b and energy $E = m_e v^2/2$ with respect to an electron bound to an atom situated at the origin of the coordinate system. Equation (3.8) gives us $\delta v_\perp$, from which we can calculate $\delta\varepsilon = m_e \delta v_\perp^2/2$. We have

$$\delta\varepsilon(E, b) = \frac{q_e^4}{Eb^2} \tag{3.32}$$

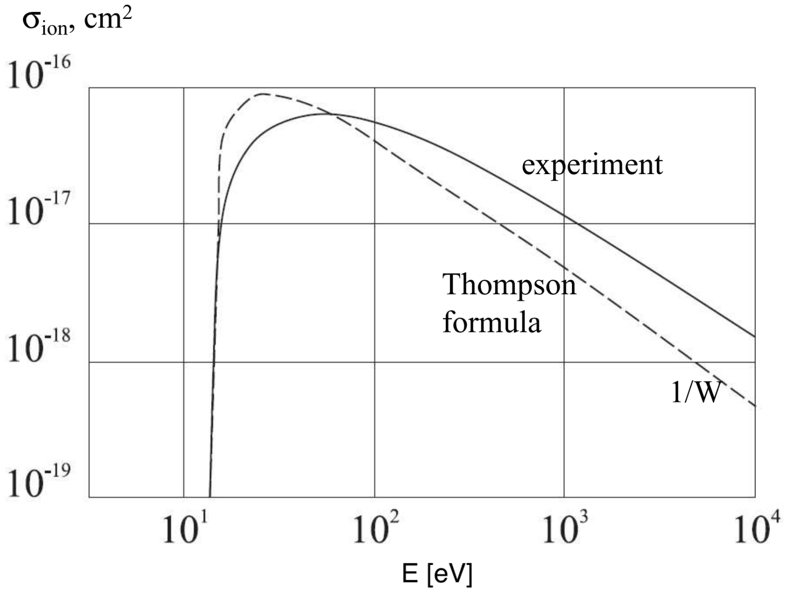


FIGURE 3.4 – Ionzation cross section as a function of the incident electron's energy and comparison of the classical formula to experiment.

We can express the differential cross section

$$d\sigma = 2\pi b db = \frac{\pi q_e^4}{E\delta\varepsilon^2} d\delta\varepsilon. \tag{3.33}$$

For ionisation to occur, we need the energy transferred $\delta\varepsilon$ to be larger than the first ionisation energy W. On the other hand, the energy transferred cannot be larger than the initial energy E of the electron. The ionisation cross section is obtained by integrating the differential cross-section between these to energies,

$$\sigma_I = \frac{\pi q_e^4}{E} \int_W^E \frac{d\delta\varepsilon}{\delta\varepsilon^2} = \pi q_e^4 \frac{E-W}{E^2 W} = \frac{\pi q_e^4}{W^2} \left(\frac{W}{E} - \frac{W^2}{E^2} \right) \tag{3.34}$$

which is the Thomson formula and is illustrated on Fig.3.4.

3.3.2 Exercise : Slowing down of a fast ion by cold electrons

The problem of angular scattering of the electrons treated in the previous section is complementary to the problem a fast ion travelling in a cold electron population. The ion, through its motion, will induce small changes in the electron velocities, and will therefore transfer some of its energy to the electron population. We assume that the ion moves at a constant velocity $\mathbf{v}_{ion}$.

1. Calculate the energy transfer transferred from the ion to an electron, during the interaction with a single electron.

2. Calculate the energy transfer per unit time, integrating on all electrons.
3. What is the slowing down timescale ν_{ie} ? Compare to the electron scattering frequency ν_{ei}.

3.3.3 Exercise : ambipolar electric field and Dreicer limit

Consider a gravitationally confined plasma. We have seen that an ambipolar electric field must exist to ensure quasi-neutrality in such a plasma. Under what conditions would this electric field be larger than the Dreicer field ? What would happen then ?

Chapter 4: Charging of a macroscopic object in a plasma

A plasma is composed of charged particles, animated by a thermal motion. An macroscopic object immersed in a plasma will, as a consequence, collect thermal electric currents onto its surfaces. So, the object will charge itself, and reach some electric potential, that will in turn modify the values of the currents.

The total charge Q of the object is determined by the equation

$$\frac{dQ}{dt} = I_e + I_i + I_{ph} + I_{sec} + ... \tag{4.1}$$

where the right-hand side sums the different current resulting from different processes (from left to right : electron current, ion current, photo-electron emission current, secondary electron emission current, others...). Note that the orientation of the currents is chosen positive for currents ongoing to the object. The equilibrium charge will be determined by the condition $dQ/dt = 0$; so, it is reached when all the currents on the object cancel each other.

4.1 Expression for the currents

We develop here a simplified but practical model for the electric currents. We assume that the object's surface is at a position $z = 0$ and is infinite along the directions x and y of a cartesian frame. We assume that the object has a potential φ, and introduce the subscript $\alpha = i, e$ to refer to a plasma population.

4.1.1 Plasma currents

If $q_\alpha \varphi < 0$, then the potential is attractive and all the particles can reach the surface. Assuming that the velocity distribution of the specie α is a Maxwellian, the current onto the surface is given by

$$I_\alpha = q_\alpha S \int_0^\infty \frac{n_\alpha}{\sqrt{2\pi} v_{th,\alpha}} e^{-v_z^2/2v_{th,\alpha}^2} v_z dv_z = I_{\alpha,0} \tag{4.2}$$

where

$$I_{\alpha,0} = q_\alpha n_\alpha v_\alpha S \tag{4.3}$$

here we introduced $v_\alpha = (kT_\alpha/2\pi m_\alpha)^{1/2}$, S is the surface of the object and the thermal velocity is $v_{th,\alpha} = \sqrt{kT_\alpha/m_\alpha}$).

Now if $q_\alpha\varphi > 0$, the potential is repulsive and only the particles having a z component of their velocity vector larger than $\sqrt{2q_\alpha\varphi/m_\alpha}$ can reach the surface of the object. The expression of the current onto the surface S is

$$I_\alpha(\varphi) = q_\alpha S \int_{\sqrt{2q_\alpha\varphi/m_\alpha}}^{\infty} \frac{n_\alpha}{\sqrt{2\pi} v_{th,\alpha}} e^{-v_z^2/2v_{th,\alpha}^2} v_z dv_z = I_{\alpha,0} \exp\left(-\frac{q_\alpha\varphi}{kT_\alpha}\right) \tag{4.4}$$

Note that to obtain these expressions, we neglected a possible drift velocity of the charged population with respect to the surface, and consider only the thermal motion. This is nearly always a very good approximation for electrons, but in general not true for ions, for two reasons :

— In space environments, ions flow will in general be supersonic (the solar wind ion fluid is supersonic, and orbital velocities of spacecraft for instance in the ionosphere are in general quite larger than the ion thermal speeds). The ion speed to consider in the expression of the current is then $v_i = u_i$, the drift speed of ions with respect to the surface.

— It can be shown that the ions (if the plasma flow to the surface is initially subsonic, typically what happens in laboratory electrostatic discharges) must enter the sheath at the Bohm speed[1] $v_i = v_B = \sqrt{kT_e/m_i}$ in order for a stable, steady-state sheath to be maintained.

4.1.2 Photoelectron and secondary electron currents

Some charging currents do not originate from the plasma, but from the charged surface itself. It is the case when the surface is illuminated by ionizing radiation, and emits photo-electrons or secondary electrons. Here, we discuss the case of photoelectrons and use the subscript "ph", but the discussion and expressions are exactly the same for emissions of secondary electrons.

In the case of a repulsive potential $\varphi < 0$, all of the electrons will leave the surface and the current will be

$$I_{ph} = eS_{lit}J_{ph,0} \tag{4.5}$$

where $J_{ph,0}$ is the photo electron flux at zero potential, which depends only on the surface illumination and on the physical properties of the illuminated material. S_{lit} is the surface receiving the UV light. Mind the sign of the expression, which comes from our

1. This is called the Bohm sheath criterion

convention of orientation of the currents, cf. eq.(4.1).

If $\varphi > 0$, the potential is attractive and only the fastest electrons will leave the surface. Others will be attracted back and not contribute to the escaping current. We have

$$I_{ph}(\varphi) = eS_{lit}J_{ph,0}\int_{\sqrt{2e\varphi/m_e}}^{\infty}\frac{1}{\sqrt{2\pi}v_{th,ph}}e^{-v_z^2/2v_{th,ph}^2}v_z dv_z = eS_{lit}J_{ph,0}\exp\left(-\frac{e\varphi}{kT_{ph}}\right). \tag{4.6}$$

The photoelectron distribution has been assumed to be Maxwellian with a temperature T_{ph}. Note that in reality, emitted photo or secondary electrons are usually not well modeled by a single Maxwellian, and multi-temperature models are often used.

4.2 Expressions of the floating potential

4.2.1 In a plasma : electron and ion current only

We consider the case were only the two first terms in the right-hand side of eq.(4.1) are of importance. We can see that, because of the small electron mass, we shall in general have $I_{e,0} \gg I_{i,0}$, and the object will tend to charge negatively. Therefore all protons will be collected (the ion current is then independent of the value of the potential, and is for this reason usually called the ion "saturation current"), whereas the electron current will depend on φ.

The potential reached in equilibrium is obtained from the condition $dQ/dt = 0$,

$$\varphi_{eq} = \frac{kT_e}{e}\ln\left(\frac{v_i}{v_e}\right) = -\frac{kT_e}{2e}\ln\left(\frac{m_i}{2\pi m_e}\right) \tag{4.7}$$

where the last equality assumes an ion current given by the Bohm criterion (typical case for laboratory measurements). In the case of a supersonic flow, v_i must be replaced by u_i and the ion mass does not appear in the expression any more.

The object then carries an equilibrium charge $Q_{eq} = C\varphi_{eq}$, with C the capacitance of the object. The equilibrium potential is then a few times the plasma electron temperature.

4.2.2 A sunlit surface : objects in the interplanetary space

In the vicinity of the Sun, objects receive ionizing solar UV. For an order of magnitude at 1 AU, $J_{ph,0} \sim 50\ \mu\text{A/m}^2$, which is much larger than the typical electron current from the interplanetary "solar wind" plasma onto the object (which is $I_{e,0}/S \sim 0.5\ \mu\text{A/m}^2$ at 1 AU). Therefore in typical interplanetary conditions, an object is charged positively.

We can obtain the equilibrium potential of, say, a spacecraft in the interplanetary space by using eq.(4.1), and neglecting the ion current

$$\varphi_{eq} = \frac{kT_{ph}}{e} \ln \left(\frac{J_{ph,0} S_{lit}}{e n_e v_e S} \right) \tag{4.8}$$

which is a few times the photo-electron temperature expressed in eV. For typical solar wind conditions, $\varphi_{eq} \sim 5 - 10$ V. Interestingly this value does not depend much on the distance from the Sun, since J_{ph_0} and n_e both vary as the inverse square of the distance from the Sun, so their ratio is approximately a constant.

4.2.3 Charge of a dust grain in the interplanetary medium

This has interesting consequences for the physics of dust grains in the interplanetary medium. The charge of a dust of size a is $q \sim C\varphi_{eq}$, with φ_{eq} given by eq.(4.8) and $C \simeq 4\pi\varepsilon_0 a$ being the capacitance of a sphere of radius a – which is a good approximation since a is much smaller than the local Debye length (which is of the order of 10 m in the solar wind at 1 AU).

Therefore the charge carried by a dust grain varies linearly with is size, $Q \sim a$. On the other hand, the mass of a dust grain is proportional to its volume, so the charge on mass ratio of a grain is inversely proportional to the square of its size, $Q/m \sim a^{-2}$.

This has important consequences on the interplanetary dust cloud dynamics : small dust grain have an important charge on mass ratio, and their dynamics will be strongly influenced by the Lorentz force (they will behave as very heavy ions) whereas large grains will be influenced by gravitational force only and have roughly Keplerian orbits.

4.3 Principle of the Langmuir probe

We can now understand the working principle of an important plasma sounding device : the Langmuir probe. The idea is to place a conducting device in a plasma and to bias it at some potential Φ_B. Measuring the intensity $I(\Phi_B)$ flowing through the device will let us estimate the density and temperature of the plasma.

In a plasma, without photoelectron emission effect, the characteristic curve looks like the one presented in Fig.4.1, and can be interpreted as follows (the orientation of the current is toward the plasma) :

— At $\Phi_B \to -\infty$, the collected current is almost completely ionic and is the ion saturation current $I_{Si} = -n_0 e v_i$, where v_i is the ion velocity at the entrance of the sheath. In a steady laboratory plasma experiment (which the fig.4.1 illustrates),

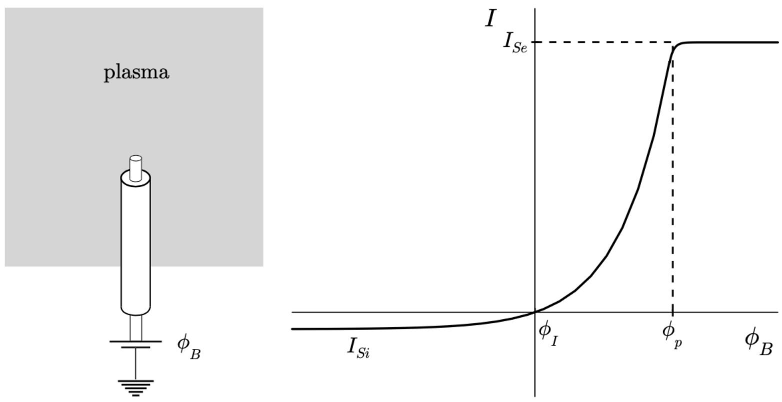


FIGURE 4.1 – Typical characteristic curve of a Langmuir probe, showing the current I drawn from the probe as a function of the applied bias voltage Φ_B. Labels are Φ_f, floating potential ; Φ_p, plasma potential – in our model, $\Phi_p = 0$; I_{Si}, ion saturation current ; I_{Se}, electron saturation current. From *J.D. Callen, Fundamentals of plasma physics, 2003.* Note that the convention for the orientation of the current is upward (to the plasma, opposite from the convention of the first section of this chapter).

it is given by $v_i^2 = kT_e/m_i$.

— At $\Phi_B \to +\infty$, the collected current is almost completely electronic. It is the electron saturation current $I_{Se} = n_0 e v_e$, where v_e is typically the thermal agitation speed, $v_e^2 = kT_e/(2\pi m_e)$.

— At $I = 0$, the potential is by definition the floating potential, that the probe would have if let passively in the plasma, $\Phi_f = -(kT_e/2e) \ln m_i/2\pi m_e$ (using the Bohm sheath criterion).

— In the region $\Phi_B < 0$ ($\Phi_B < \Phi_p$ on fig.4.1), the current varies exponentially with the applied voltage, since $I_e \propto \exp(-e\Phi_B/kT_e)$. Plotting this part in log-log then makes it possible to determine robustly the electron temperature.

So, the characteristic makes it possible to determine independently and robustly determine the plasma temperature and the plasma density at infinity from the probe n_0.

4.4 Exercise and examples

4.4.1 Exercise : levitation of lunar dust

When illuminated by the Sun, the lunar surface charges positively under the action of photoelectron emission. A photoelectron sheath is present above the surface. Its density

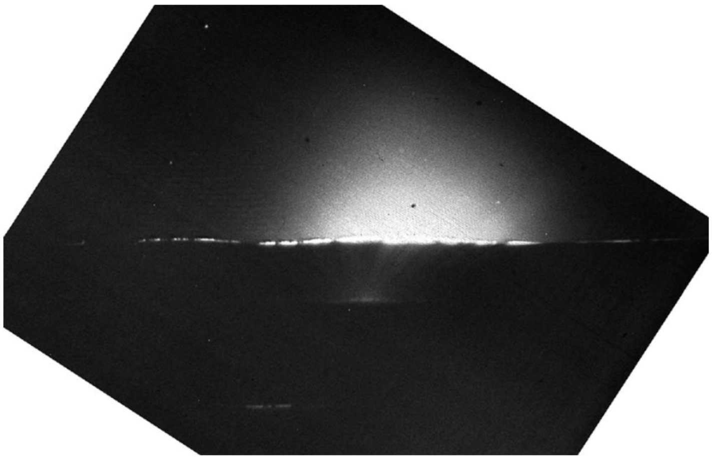


Fig. 1. Surveyor 6 image 328141526.354 showing a glow on the western lunar horizon after sunset. The broad and high diffuse glow is zodiacal light from interplanetary dust. The low bright band just at the horizon is lunar "Horizon Glow" apparently due to light scattered from dust particles near the lunar surface. National Space Science Data Center.

FIGURE 4.2 – From *Colwell et al, 2009*

is given by

$$n_{pe}(z) = n_{pe0}\left(1 + \frac{z}{\sqrt{2}\lambda_D}\right)^{-2} \tag{4.9}$$

where z is the altitude, n_{pe0} the density at the ground level, and λ_D is the photoelecton Debye length calculated with the density n_{pe0} and a temperature $T_{ph} \sim 3$ eV.

Calculate the altitude at which a dust grain levitates, as a function of its typical radius r.

4.5 The Bohm sheath criterion*

In chapter 2, we made a model of the plasma sheath next to a charged surface. Although practical to highlight the role of the Debye length, this model happened to not be completely accurate, because of an inappropriate modeling of the ions dynamics. Assume the object is charged negatively. The electrons or ions dynamics in a steady state is given by

$$n_\alpha u_\alpha \frac{du_\alpha}{dz} = kT_\alpha \frac{dn}{dz} - n_\alpha q_\alpha \frac{d\varphi}{dz} \tag{4.10}$$

The ratio of the the macroscopic kinetic energy term to the pressure term is of the order of the square of the Mach number $\mathrm{Ma}^2 = u^2/v_{th,\alpha}^2$, where the directed kinetic eneregy that a particle can acquire is of the order of the electrostatic potential drop $\varphi \sim kT_e$. For electrons, both terms are of the same order of magnitude – for the sake of simplicity, we will drop the kinetic energy term and assume that the electron density is given by the Boltzmann law

$$n_e(z) = n_\infty e^{e\varphi(z)/kT_e}. \tag{4.11}$$

On the other hand, because of the ion to electron mass ratio, the ion square mach number is by 3 orders of magnitude larger than the pressure term : our previous modeling of the ions as in Boltzmann equilibrium is not adequate, and we must instead calculate its density from the dynamics equation,

$$\frac{1}{2}m_i u_i(z)^2 + e\varphi(z) = \frac{1}{2}m_i u_i(\infty)^2 \Rightarrow u_i(z) = \sqrt{u_i(\infty)^2 - \frac{2e\varphi(z)}{m_i}} \tag{4.12}$$

together with the continuity equation

$$n_i(z)u_i(z) = n_\infty u_i(\infty) \Rightarrow n_i(z) = \frac{n_\infty}{\sqrt{1 - 2e\varphi(z)/(m_i u_i(\infty)^2)}}. \tag{4.13}$$

The potential in the sheath is thus given by the Poisson equation, but with the ion density given by eq.(4.14) instead of the Boltzmann formula. The resulting equation is strongly non-linear and no analytical solution can be found. Numerical solutions can be used for a proper modeling of the sheath. But in order to get a qualitative modeling of the sheath, we may just linearize the equation by assuming that $e\varphi \ll kT_e, m_i u_i(\infty)^2$. The linearized Poisson equation is

$$\frac{d^2\varphi}{dz^2} = \frac{1}{\lambda_{D,e}^2}\left(1 - \frac{kT_e}{m_i u_i(\infty)^2}\right)\varphi \tag{4.14}$$

where $\lambda_{D,e}^2 = \varepsilon_0 kT_e/n_\infty e^2$. If the parenthesis in the right hand side is negative, then we have for solution an harmonic oscillator : this is incompatible with our boundary conditions implying a steady plasma at infinity. So, the plasma must somehow organize itself so that the ions velocity at the entrance of the sheath verifies Bohm's criterion

$$u_i > \sqrt{\frac{kT_e}{m_i}}, \tag{4.15}$$

and the right hand side is sometimes called Bohm's velocity. In fact, the criterion is usually just fulfilled, and for practical cases it is possible to assume that ions practically enter the sheath with Bohm's speed.

Chapter 5: Motion of a charged particle in electromagnetic fields

We have seen that plasma systems are characterized by a strong coupling between the dynamics of the particles and those of the electromagnetic field. To understand a plasma, it is therefore of fundamental importance to have a clear understanding of the motion of charged particles in prescribed electromagnetic fields. This part introduces the main aspects of particle motions in given fields.

In whole of this section, $\mathbf{B} = B\mathbf{b}$ is the magnetic field vector. B is its modulus and $\mathbf{b}$ the unit vector along the magnetic field line. The parallel component of a vector is its scalar product with $\mathbf{b}$, for instance $v_{\|} = \mathbf{v} \cdot \mathbf{b}$. Its perpendicular component is the remaining, $\mathbf{v}_{\perp} = \mathbf{v} - v_{\|}\mathbf{b}$.

5.1 Charged particle in constant fields

We start by the most simple case, where the magnetic and electric fields are constant in time and space. So we have here $B = const.$, and $\mathbf{b} = \mathbf{const.}$.

5.1.1 The cyclotron motion

In the absence of an electric field, the Lorentz force acting on a particle of charge q and mass m is $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. This force does not produce any work on the particle, since $\mathbf{F} \cdot \mathbf{v} = 0$ all the time. The kinetic energy of a particle in purely magnetic field is thus a constant of the motion.

Separating the equation of motion between parallel and perpendicular components we obtain

$$\begin{cases} \dot{v}_{\|} = 0 \\ \dot{\mathbf{v}}_{\perp} = \omega_c \mathbf{v}_{\perp} \times \mathbf{b}. \end{cases} \tag{5.1}$$

where we introduced the <u>*gyro-frequency*</u> of the particle [1] $\omega_c = qB/m$. Thus, the parallel component of the particle's velocity is a constant of the motion, and the norm of its perpendicular component is another one. Introducing two cartesian axis (x, y) in the

1. Note that ω_c is an algebraic quantity, that can be positive or negative depending on the sign of q.

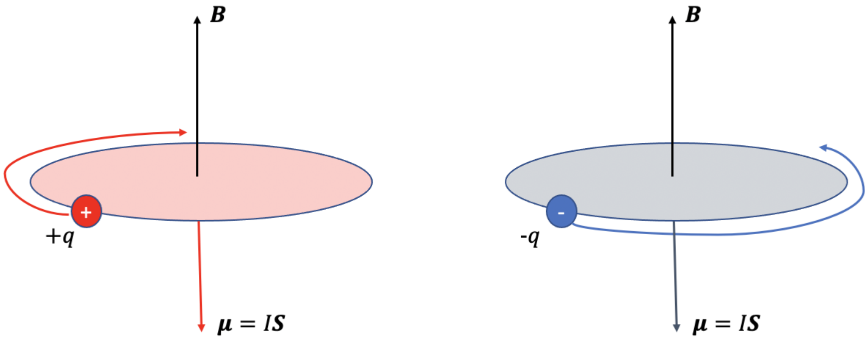


FIGURE 5.1 – Cyclotron motion of positively and negatively charged particles. The current $I \propto qv$ is oriented in the same direction in both cases, hence the anti-parallel direction of the magnetic moment, independent of the particle's charge (cf. sec.5.1.2).

plane perpendicular to $\mathbf{b}$, and using the complex notations $\bar{v}_\perp = v_x + iv_y$ we obtain

$$\dot{\bar{v}}_\perp = -i\omega_c \bar{v}_\perp \Rightarrow \bar{v}_\perp = \bar{v}_\perp(0)e^{-i\omega_c t} \tag{5.2}$$

and the trajectory of the particle is given by

$$\dot{\bar{r}}_\perp = \bar{v}_\perp \Rightarrow \bar{r}_\perp = \bar{r}_\perp(0) + \frac{i\bar{v}_\perp(0)}{\omega_c}\left(e^{-i\omega_c t} - 1\right). \tag{5.3}$$

So, the particle describes a circle in the perpendicular plane. The radius of this circle is $\rho_\ell = |v_\perp/\omega_c|$, and is called the *Larmor radius* of the particle (and is a positive quantity). A particle with a positive charge describes a clockwise circle around $\mathbf{b}$ (right-hand polarization), whereas a negative charge describes an anti-clockwise circle (left-hand polarization with respect to the magnetic field).

A bit more useful vocabulary : the angle of the velocity vector with respect to the magnetic field line is called the *pitch-angle* θ,

$$\theta = \arccos\frac{v_\parallel}{v} = \arctan\frac{v_\perp}{v_\parallel} \tag{5.4}$$

The rotation phase of the particle in the perpendicular plane is called the *gyro-phase*,

$$\varphi(t) = \omega_c t + \arg i\bar{v}_\perp(0) \tag{5.5}$$

The gyrophase-averaged position of a particle is called its *guiding center* position $\mathbf{R}_g$. In the present case,

$$\mathbf{R}_g(t) = \mathbf{R}_g(0) + v_\parallel t\mathbf{b}, \tag{5.6}$$

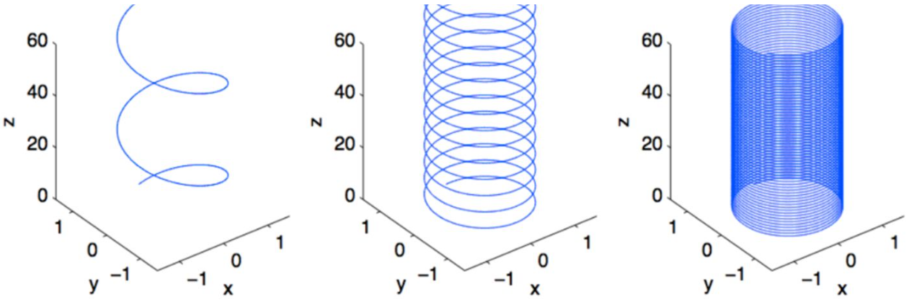


FIGURE 5.2 – Trajectories of a positively charged particle in a constant magnetic field directed along the z axis, for three different values of its pitch angle ($\theta = 10°, 45°, 80°$ from left to right. Distances are normalized by the particle's Larmor radius.

the guiding center follows the straight magnetic field line, with a constant velocity.

In relativistic regime, the equation of motion is $\dot{\mathbf{p}} = q\mathbf{v} \times \mathbf{B}$ with $\mathbf{p} = \gamma m \mathbf{v}$ and $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor. Since $v = const.$, the Lorentz factor is a constant as well and the analysis made above holds, with the change $\omega_c = qB/\gamma m$. The Larmor radius is $\rho_\ell = \gamma m v_\perp / qB = p_\perp / qB$ with p the Lorentz-invariant momentum.

A last point of vocabulary. In the context of particle physics, the momentum of a particle in a magnetic field is often described through its rigidity, which is counted in Volts and defined as

$$R = \rho_\ell c B = \frac{p_\perp c}{|q|} \tag{5.7}$$

Useful orders of magnitudes are :

— Electron gyro-frequency

$$\omega_{c,e}\ [\text{rad.s}^{-1}] \simeq 176 \frac{B[\text{nT}]}{\gamma} \qquad f_{c,e}\ [\text{Hz}] \simeq 28 \frac{B[\text{nT}]}{\gamma} \tag{5.8}$$

— Ion gyrofrequency (atomic number Z, mass number A)

$$\omega_{c,i}\ [\text{rad.s}^{-1}] \simeq 10^{-1} \frac{Z}{A} \frac{B[\text{nT}]}{\gamma} \qquad f_{c,i}\ [\text{Hz}] \simeq 1.5 \times 10^{-2} \frac{Z}{A} \frac{B[\text{nT}]}{\gamma} \tag{5.9}$$

— Larmor radius (non-relativistic limit, $\mathscr{E}_\perp = p_\perp^2/2m = mv_\perp^2/2$)

$$\rho_{\ell,e}\ [\text{km}] \simeq 3.4 \frac{\sqrt{\mathscr{E}_\perp}\ [\text{eV}]}{B\ [\text{nT}]} \qquad \rho_{\ell,i}\ [\text{km}] \simeq 144 \frac{\sqrt{\mathscr{E}_\perp}\ [\text{eV}]}{B\ [\text{nT}]} \times \frac{\sqrt{A}}{Z} \tag{5.10}$$

where $\mathscr{E}_\perp$ is the perpendicular kinetic energy of the particle.

— Larmor radius (ultra-relativistic limit, $\mathscr{E}_\perp = p_\perp c$)

$$\rho_\ell \text{ [A.U.]} \simeq 2 \times 10^{-2} \frac{\mathscr{E}_\perp \text{ [GeV]}}{B \text{ [nT]}} \equiv 2 \times 10^{-2} \frac{R \text{ [GV]}}{B \text{ [nT]}} \quad (5.11)$$

5.1.2 Plasma diamagnetism

Charged particles in a field $\mathbf{B}_0$ rotate in such a way to produce a current that in turn generates a magnetic field $\delta\mathbf{B}$ that opposes $\mathbf{B}_0$. So, the plasma is a diamagnetic medium. The microscopic magnetic moment $\boldsymbol{\mu}$ associated to the current loop of a gyrating particle is

$$\boldsymbol{\mu} = I\mathbf{S} = -\frac{q\omega_c}{2\pi}\pi\rho_\ell^2\mathbf{b} = -\frac{\mathscr{E}_\perp}{B}\mathbf{b}. \quad (5.12)$$

It is independent of the particle's charge, since $q\omega_c > 0$ whatever is the sign of the particle. The magnetization vector of the plasma is the volumetric density of magnetic moment, which, in a plasma of density n, is

$$\mathbf{M} = 2n\boldsymbol{\mu} = -\frac{2nkT_\perp}{B}\mathbf{b}, \quad (5.13)$$

if we assume that the electrons and ions have the same temperature $kT_\perp = \langle \mathscr{E}_\perp \rangle$. Note that the plasma is not a linear medium.

Consider the following situation : a system of currents $\mathbf{j}_{ext}$, external to the plasma (for example circulating in a solenoid), produces a magnetic field $\mathbf{B}_0$ in the plasma. What decrease in the magnetic field inside the plasma will be produced by the plasma pressure ? Using Ampère's law, which in the magnetized medium, is

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{j}_{ext} + \boldsymbol{\nabla} \times \mathbf{M}\right) = \boldsymbol{\nabla} \times \left(\mathbf{B}_0 + \mu_0\mathbf{M}\right) \quad (5.14)$$

we have in the plasma

$$\mathbf{B} = \mathbf{B}_0 + \mu_0\mathbf{M} \simeq \left(1 - \frac{nkT_\perp}{B_0^2/2\mu_0}\right)\mathbf{B}_0 \simeq (1-\beta)\mathbf{B}_0 \quad (5.15)$$

where we have introduced the dimensionless *plasma* β parameter, equal to the ratio of the plasma pressure to the magnetic field pressure. Strictly speaking, our expression is valid only in the limit of very small values of β (since the magnetization has been calculated from the external field $\mathbf{B}_0$ and not self-consistently from the plasma magnetic field $\mathbf{B}$). This expression

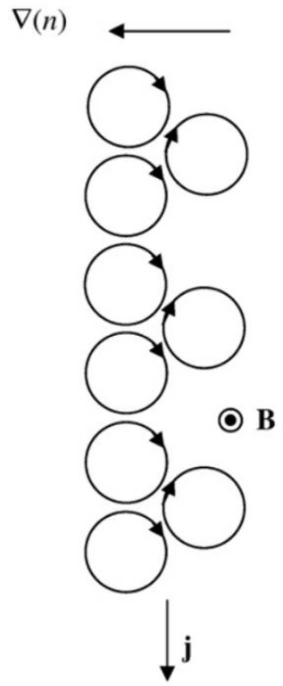


FIGURE 5.3 – Illustration of the origin of the plasma magnetisation current.

shows that the thermal pressure decreases the value of the external magnetic field inside the plasma.

The current density associated to this magnetization appearing in Ampère's law is

$$\mathbf{j}_{mag} = \boldsymbol{\nabla} \times \mathbf{M} = -\frac{\boldsymbol{\nabla}(2nkT_\perp) \times \mathbf{b}}{B} \tag{5.16}$$

and is called the magnetization current. It is perpendicular both to the magnetic field and to the pressure gradient. This current is not associated to a physical displacement of charged particles in the volume of the plasma, but results from the non-compensation of the currents carried by the Larmor rotation of the particles when a pressure gradient exists.

5.1.3 Constant electric field

In the presence of a constant electric field, the equation of motion of the charged particle are now

$$\begin{cases} \dot{v}_\parallel = qE_\parallel/m \\ \dot{\mathbf{v}}_\perp = \omega_c \mathbf{v}_\perp \times \mathbf{b} + q\mathbf{E}_\perp/m. \end{cases} \tag{5.17}$$

Therefore, the motion along the magnetic field is uniformly accelerated, just as it would be in the absence of the magnetic field,

$$v_\parallel(t) = v_\parallel(0) + \frac{qE_\parallel t}{m} \qquad r_\parallel(t) = r_\parallel(0) + v_\parallel(0)t + \frac{qE_\parallel t^2}{2m} \tag{5.18}$$

The motion in the perpendicular plane consists of two components. The first is given by the solution of the homogeneous equation, and correspond to the cyclotron motion, as studied in the begnining of this section – cf. eqs.(5.2)-(5.3) . The second component is given by a particular solution to the differential equation. A trivial solution is the one with constant velocity $\dot{\mathbf{v}}_{\perp,p} = 0$,

$$\mathbf{v}_{\perp,p} \times \mathbf{b} = -\frac{q\mathbf{E}_\perp}{m\omega_c} \Rightarrow \mathbf{v}_{\perp,p} = \frac{\mathbf{E}_\perp \times \mathbf{b}}{B}. \tag{5.19}$$

This constant perpendicular velocity appearing in the presence of an electric field is called the "E cross B" drift, or "cross field drift" velocity and plays a very important role in plasma physics.

$$\mathbf{v}_\times = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{5.20}$$

Importantly, this drift does not depend on the charge nor on the mass of the particles : under the action of a constant electric field, the plasma drifts as a whole in the direction both perpendicular to $\mathbf{E}$ and $\mathbf{B}$.

A direct and important interpretation of this drift comes the transformation of the electric field by a change of frame of reference. In the non-relativistic limit, the change is $\mathbf{E}' = \mathbf{E} + \mathbf{u}_{R'/R} \times \mathbf{B}$, while the magnetic field is invariant by a Galilean change of frame. Therefore, it is always possible to find a frame in which the electric field vanishes, and the motion of the charged particle consists in the pure cyclotron motion. The velocity of this specific frame of reference checks

$$\mathbf{u}_{R'/R} \times \mathbf{B} = -\mathbf{E} \Rightarrow \mathbf{u}_{R'/R,\perp} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} = \mathbf{v}_{\times} \tag{5.21}$$

so, its perpendicular component is just the cross-field drift velocity (its parallel component is undetermined and can be anything). The velocity of the particle in the frame R is then the superposition of the cyclotron motion, which is the only motion in R', and the motion of the frame R' with respect to R. Therefore, one can think of the cross field velocity $\mathbf{v}_{\times} = \mathbf{u}_{R'/R}$ as the velocity of the magnetic field lines themselves.

Exercise : Plasma in a solenoid. One translates the solenoid at a speed V. What is the dynamics of the plasma ?

5.1.4 Constant force field

Under the action of an homogeneous force field $\mathbf{F}$, the analysis performed in the specific case of the electric field still holds. One just have to replace $\mathbf{E}$ by $\mathbf{F}/q$, and obtain the generic expression for the force drift

$$\mathbf{v}_F = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} \tag{5.22}$$

this force drift depends on the charge of the particle (if $\mathbf{F}$ does not depend linearly on q). Therefore, ions and electrons will in general drift in opposite directions, producing a drift current, and a polarization of the plasma. Gravitational and inertial forces are, for this reasons, responsible for the appearance of plasma currents.

5.2 Motion of particles in inhomogeneous fields

In this section, we consider the motion of particles in fields that can vary in space and time, under the assumption that the variation of the fields are small on time and length scales associated with the cyclotron motion : $\rho_\ell \cdot \boldsymbol{\nabla} \ll 1$

5.2.1 Guiding center motion : general equations

We separate the particle motion into its cyclotron motion, and the guiding center motion :

$$\mathbf{r}(t) = \mathbf{R}_g(t) + \mathbf{r}_\ell(t), \qquad \mathbf{v}(t) = \mathbf{V}_g(t) + \mathbf{v}_\ell(t), \tag{5.23}$$

where $\mathbf{v}_\ell(t) = \dot{\mathbf{r}}_\ell(t)$ is perpendicular to $\mathbf{b}$ and given by eq.(5.2). Averaging over the cyclotron motion, we have $\langle \mathbf{r}(t) \rangle = \mathbf{R}_g(t)$, and $\langle \mathbf{v}(t) \rangle = \mathbf{V}_g(t)$.

The motion of a particle in the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ is described by the equation

$$\frac{d\mathbf{V}_g}{dt} + \frac{d\mathbf{v}_\ell}{dt} = \frac{q}{m} \left(\mathbf{E}(\mathbf{R}_g) + \mathbf{r}_\ell \cdot \boldsymbol{\nabla} \mathbf{E}(\mathbf{R}_g) + (\mathbf{V}_g + \mathbf{v}_\ell) \times (\mathbf{B}(\mathbf{R}_g) + \mathbf{r}_\ell \cdot \boldsymbol{\nabla} \mathbf{B}(\mathbf{R}_g)) \right). \tag{5.24}$$

where we made a linear approximation of the fields on the scale ρ_ℓ. Averaging over the cyclotron motion and retaining only the zeroth order terms, we have,

$$\mathbf{E}(\mathbf{R}_g) + \mathbf{V}_g^{(0)} \times \mathbf{B}(\mathbf{R}_g) = 0 \Rightarrow \mathbf{V}_g^{(0)} = \frac{\mathbf{E}(\mathbf{R}_g) \times \mathbf{B}(\mathbf{R}_g)}{B(\mathbf{R}_g)^2} = \mathbf{v}_\times(\mathbf{R}_g) \tag{5.25}$$

where $d\mathbf{V}_g^{(0)}/dt$ has been assumed a first order term, and where the electric field has been assumed to have no parallel component [2]. The perpendicular motion, at zeroth order, just consists in the cross field drift calculated from the value of the fields at the particle's guiding center position. So, at the order 0, the guiding center motion is exactly the one that was described previously in the case of homogeneous field, which was to be expected, since at the order 0, the fields are indeed homogenous.

We now go to the order 1, where new physical effects appear, linked to the existence of gradients in the fields. After averaging over the cyclotron motion, all the terms linear in $\mathbf{v}_\ell$ and $\mathbf{r}_\ell$ vanish and we have

$$\frac{d\mathbf{V}_g^{(0)}}{dt} = \frac{q}{m} \left(\mathbf{V}_g^{(1)} \times \mathbf{B}(\mathbf{R}_g) + \langle \mathbf{v}_\ell \times \mathbf{r}_\ell \cdot \boldsymbol{\nabla} \mathbf{B}(\mathbf{R}_g) \rangle \right). \tag{5.26}$$

The term in bracket does not average out to zero, since it is quadratic in $\mathbf{v}_\ell$ and $\mathbf{r}_\ell$. We can show that

$$\langle \mathbf{v}_\ell \times \mathbf{r}_\ell \cdot \boldsymbol{\nabla} \mathbf{B}(\mathbf{R}_g) \rangle = -\frac{v_\perp^2}{2\omega_c} \boldsymbol{\nabla} B = -\frac{\mu}{q} \boldsymbol{\nabla} B \tag{5.27}$$

We write the term $\mathbf{V}_g^{(0)} = v_\parallel \mathbf{b} + \mathbf{v}_\times$. So, its time derivative can be written as

$$\frac{d\mathbf{V}_g^{(0)}}{dt} = \frac{dv_\parallel}{dt} \mathbf{b} + v_\parallel \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{v}_\times}{dt} \tag{5.28}$$

These three last equations constitute the basis to study the motion of the guiding center in inhomogeneous fields. Note that the electric field inhomogeneity plays no role in this motion [3]. In the next paragraph, we first investigate the motion in the parallel direction, and then will look at the perpendicular drifts.

2. In the presence of a parallel electric field, the parallel motion is just given by $dV_{g,\parallel}/dt = qE_\parallel/m$
3. Actually, it intervenes as a second order effect, involving the second space derivative of $\mathbf{E}(\mathbf{r})$

5.2.2 Magnetic moment conservation, and mirror force

Taking the scalar product of eq.(5.26) with **b**, we obtain

$$m\frac{dv_\parallel}{dt} = -\mu\mathbf{b}\cdot\boldsymbol{\nabla}B \tag{5.29}$$

So, the right hand term plays the role of a pseudo-force, modifying the parallel component of the velocity in the presence of converging or diverging field lines. It is called the magnetic mirror force.

As a consequence of eq.(5.29), the magnetic moment μ of the particle is a constant of the particle motion. Indeed, the variation of the parallel kinetic energy is

$$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu v_\parallel\mathbf{b}\cdot\boldsymbol{\nabla}B = -\mu\frac{dB}{dt} \tag{5.30}$$

where we have assumed that the magnetic field does not vary with time ($\partial_t B = 0$). Since the effect does not involve any electric field, it lets the particle kinetic energy constant. We must have

$$\frac{d}{dt}\left(\mathscr{E}_\perp + \mathscr{E}_\parallel\right) = \frac{d}{dt}(\mu B) - \mu\frac{dB}{dt} = B\frac{d\mu}{dt} = 0 \tag{5.31}$$

since $B \neq 0$, the magnetic moment μ must be conserved along the particle's trajectory. It is in fact an adiabatic invariant of the particle's motion, valid even for intrinsic time-variations of the magnetic field, given these happen on timescales much slower than ω_c^{-1}.

One may conveniently rewrite the conservation of the particle's kinetic energy as

$$\frac{1}{2}mv_\parallel^2(s) + \mu B(s) = \mathscr{E} = const. \tag{5.32}$$

where s is a coordinate along a magnetic field line. Since μ is a constant, the magnetic field modulus $B(s)$ plays exactly the role of a potential energy for the parallel motion : a strong increase of the magnetic field (corresponding to a strong convergence of field lines) will reflect charged particles like mirrors.

Magnetic bottle

A magnetic field configuration with two strong convergence points is called a magnetic bottle. Such a configuration is characterized by its mirror ratio $R_m = B_{max}/B_{min}$, which characterizes its efficiency to trap particles. A particle will escape the bottle if its total kinetic energy checks $\mathscr{E} > \mu B_{max}$. If we call θ_{min} the pitch-angle of the particle at the position where the magnetic field is B_{min}, then the escape condition reads

$$\sin^2\theta_{min} < \frac{B_{min}}{B_{max}} = 1/R_m. \tag{5.33}$$

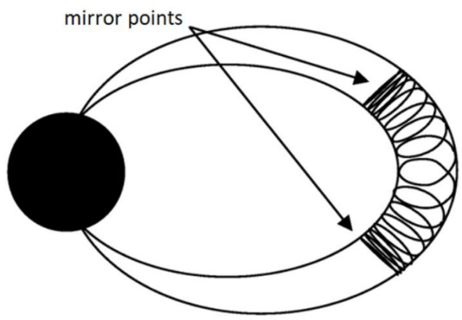


FIGURE 5.4 – Particle trajectory in the magnetic bottle produced by the converging field lines of a planetary dipole.

Eq.(5.33) defines the angle $\theta_m = \arcsin \sqrt{1/R_m}$ of aperture of a cone in velocity space, called the *loss cone*. The particles inside the loss cone will able to escape the magnetic trap. Therefore the plasma velocity distribution function inside a magnetic bottle is usually not an equilibrium Maxwellian, but rather a Maxwellian minus the loss particles (if collisions are completely neglected). Collisions (or electromagnetic instabilities) will tend to randomize the values of the particle's pitch angles, and as a consequence to send particles inside the loss cone, producing a leak in the bottle.

Conservation of magnetic flux through the particle ring current

A consequence of the mirror force is that the flux of the magnetic field through a surface resting on the contour defined by the Larmor radius of the particle is always conserved. This can be easily seen from the expression of this flux

$$\Phi_B = \iint \mathbf{B} \cdot d\mathbf{S} \simeq B(s)\pi\rho(s)^2 = \frac{\pi m^2}{q^2}\frac{v_\perp^2}{B} = \frac{2\pi m}{q^2}\mu = const. \tag{5.34}$$

This result is convenient to represent oneself the trajectory of a charged particle : the trajectory is wrapped around a magnetic flux tube. This is illustrated by fig.5.4.

5.2.3 Perpendicular drifts

We obtain the perpendicular motion of the guiding center by taking the vector product of eq.(5.26) by **b**, we obtain

$$\mathbf{V}_{g,\perp}^{(1)} = \frac{1}{\omega_c}\mathbf{b} \times \left(\frac{\mu}{m}\boldsymbol{\nabla} B + \frac{d\mathbf{V}_g^{(0)}}{dt} \right) \tag{5.35}$$

The first term in the parenthesis is called the "grad B" drift. The second term describes inertial effects due to the acceleration of the guiding center at zeroth order. Using eq.(5.28), we get

$$\mathbf{V}_g^{(1)} = \frac{1}{\omega_c}\mathbf{b} \times \left(\frac{\mu}{m}\boldsymbol{\nabla}B + v_\parallel\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{v}_\times}{dt}\right) \tag{5.36}$$

The second term is named the curvature drift, and the third term the polarization drift. We note that none of the terms in the parenthesis depend on the particle charge or mass. So, the only mass and charge dependence are contained in ω_c. This has two consequences : first, electrons and ions will drift in opposite directions, creating polarisation currents in the plasma. Second, all of these drift speeds are proportional to the mass/charge ratio, and therefore will be essentially carried by the ionic species in the plasma.

Grad-B drift

The gradient drift stems from the existence of a perpendicular gradient of the magnetic field modulus. The expression of the drift speed is

$$\mathbf{v}_\nabla = \frac{\mu}{q}\frac{\mathbf{B}\times\nabla B}{B^2} = -\frac{mv_\perp^2}{2qB}\frac{\nabla B\times\mathbf{B}}{B^2} \tag{5.37}$$

In this case the guiding center will drift in a direction both perpendicular to the field and to the gradient. Qualitatively, it can be seen as the fact that the larmor radius is slightly smaller in the region of large field than in the region of small field, making the trajectory a cycloid in the direction perpendicular to ∇B.

Curvature drift

The total derivative of $\mathbf{b}$ along the particle's trajectory is

$$\frac{d\mathbf{b}}{dt} = \partial_t\mathbf{b} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{b} = \partial_t\mathbf{b} + \mathbf{v}_\times\cdot\boldsymbol{\nabla}\mathbf{b} + v_\parallel\mathbf{b}\cdot\boldsymbol{\nabla}\mathbf{b} \tag{5.38}$$

And this inertial drift is strictly speaking composed of three terms. In practice, one nearly always have $v_\parallel \gg v_\times$ and $v_\parallel\mathbf{b}\cdot\nabla \gg \partial_t$ (i.e. the particle will perceive spatial changes in the direction of B on its way along the field line much faster than any intrinsic temporal change in the field line direction). Then, the drift is almost only due to the curvature term. Introducing the local curvature radius of the field line R_c, such that

$$\mathbf{b}\cdot\nabla\mathbf{b} = -\frac{\mathbf{n}}{R_c} \tag{5.39}$$

where $\mathbf{n}$ is the unit vector perpendicular to the trajectory (oriented outward from the center of curvature), one may conveniently express the drift as

$$\mathbf{v}_c = \frac{m}{q}\frac{\mathbf{B}\times v_\parallel^2(\mathbf{b}\cdot\nabla\mathbf{b})}{B^2} = \frac{mv_\parallel^2}{qR_c}\frac{\mathbf{n}\times\mathbf{B}}{B^2} \tag{5.40}$$

Finally, let's note that the left part of the vectorial product is the centrifugal force $\mathbf{F} = mv_\parallel^2\mathbf{n}/R_c$ applied on a particle following a curved trajectory at constant velocity $v_\parallel$. This curvature drift then appears to be the force drift eq.(5.22) associated to the centrifugal force.

Polarization drift

The last term of eq.(5.36) involve the time derivative of the cross field drift. It makes it possible to investigate the effect of the time variation of an applied perpendicular electric field $\dot{\mathbf{E}}_\perp$. The drift speed is in this case

$$\mathbf{v}_p = \frac{m}{qB^2}\frac{\mathbf{B}\times(\dot{\mathbf{E}}_\perp\times\mathbf{B})}{B^2} = \frac{m}{qB^2}\frac{d\mathbf{E}_\perp}{dt}, \tag{5.41}$$

and is called the polarization drift. This drift is (for once) parallel to the applied electric field, and produces an important contribution to the perpendicular polarisability (or dielectric response) of a plasma, hence its name. The polarisation current produced by applying an AC electric field to a plasma is (neglecting the electron contribution),

$$\mathbf{j}_p \simeq \frac{nm_i}{B^2}\frac{d\mathbf{E}_\perp}{dt} = \frac{\partial\mathbf{P}}{\partial t} \Rightarrow \mathbf{P} \simeq \frac{nm_i}{B^2}\mathbf{E}_\perp \equiv \chi_\perp\mathbf{E}_\perp \tag{5.42}$$

where P is the plasma polarisation vector, $\chi_\perp = nm_i/B^2$ the plasma polarisability and $\epsilon_\perp = \epsilon_0(1+\chi_\perp/\epsilon_0)$ the (perpendicular) dielectric constant of the magnetized plasma.

5.3 Adiabatic invariants

A periodic motion of period T is characterized by the existence of Poincaré invariants, which stay approximately constant under variations of the system parameters that are slow compared to T. These invariants take the form

$$I = \oint \mathbf{p}\cdot d\mathbf{q} \tag{5.43}$$

where $\mathbf{p}$ and $\mathbf{q}$ are conjugate dynamical variables, and the closed integral implies is performed on a full period, which indeed describes a closed curve in phase space. These invariants can also conveniently be formulated as an integral over time,

$$I = \int_{T_q} W_q(t)dt = \langle W_q(t)\rangle T_q, \tag{5.44}$$

where T_q is the period associated to the periodic motion of the coordinate q, and W_q is the energy associated to the cyclic coordinate q.

These integrals can be shown to be conserved to the first order in τ/T_q, where τ is the timescale on which the perturbation of the system is applied. The demonstration of this result can be found in any good analytical mechanics monographs.

The existence of these invariants prove very convenient to study periodic motions in general, and the motion of charged particles in magnetic field in particular. The example of magnetic bottles, or of a particle trapped in the earth magnetic field, is of interest.

5.3.1 First adiabatic invariant : the magnetic moment

We first consider the fastest periodic motion of our system, i.e. the cyclotron motion. The associated adiabatic invariant is

$$I_1 = \langle W_q(t) \rangle T_q = \frac{1}{2} m v_\perp^2 \frac{2\pi}{\omega_c} = \frac{2\pi m}{q} \mu \tag{5.45}$$

which is just the magnetic moment of the particle, to constant factor. We recover the constancy of this quantity, that we've demonstrated through the mirror force, for a time varying magnetic field.

5.3.2 Second adiabatic invariant : bounce motion

In a magnetic trap, the particle will oscillate between two mirror points defined by $\mu B(s_m) = \mathscr{E}$, as seen previously. Let's call T_b the "bounce period" associated to this motion. The associated adiabatic invariant is

$$I_2 = \langle W_s(t) \rangle T_s = \frac{1}{2} m \langle v_\parallel^2 \rangle T_b \tag{5.46}$$

So if the bouce period of the particle is varying in time (on timescales much larger than T_b), the parallel kinetic energy of the particle will vary as well. There are two main reasons why it may occur :

— Perpendicular drifts may convect to shorter field lines (closer mirror points) : then the Bounce period decreases and the mean kinetic energy of the particle will increase. Of course the opposite reasoning applies if the particle drift toward longer field lines.

— The magnetic field configuration may have some intrinsic time variation. For example the magnetic bottle may contract on itself, and the particle will be trapped between two approaching magnetic walls. It will as a consequence gain energy : this phenomena is called the first-order Fermi acceleration, and can be responsible for the production of cosmic rays of very high energies.

5.3.3 Third adiabatic invariant : enclosed magnetic flux

A third periodic motion may be identified in magnetic traps : once averaged over the bounce motion, the particle may be seen as a "magnetic shell", consisting in the magnetic flux tube bounded by its two mirror points. This magnetic shell moves perpendicularly to the field lines under the effect of the perpendicular drifts. In the example of a particle trapped in the Earth dipolar field, this motion will be azimuthal and associated to a momentum $p_\phi = m v_\phi + q A_\phi$, where $\mathbf{A}$ is the field vector potential. Then

$$I_3 = \int_0^{2\pi} p_\phi r d\phi \simeq q \oint \mathbf{A} \cdot d\boldsymbol{\ell} = q \Phi_B \sim q \pi R^2 B_0 \tag{5.47}$$

where Φ_B is the total magnetic flux enclosed by the azimuthal motion of the particle around the earth, and R the approximate radius of the particle's orbit. Thus, if, for some reason, the effective magnetic field B_0 of the Earth increases, the particle orbit will tend to diminish its radius to keep the enclosed flux constant. This invariant is not in practice very useful, because events making the total magnetic field vary (e.g. magnetic storms caused by the interaction of the magnetosphere with a coronal mass ejection) will tend to occur on timescales that are of the order, or smaller, than the periodic motion of particles around the Earth, and the adiabatic invariant is not conserved under these non-adiabatic conditions.

5.4 Examples and exercises

5.4.1 The magnetic mirror term

Show that

$$\langle \mathbf{v}_\ell \times \mathbf{r}_\ell \cdot \boldsymbol{\nabla}\mathbf{B}(\mathbf{R}_g)\rangle = -\frac{\mu}{q}\boldsymbol{\nabla}B \tag{5.48}$$

5.4.2 Time invariance of the magnetic moment

We have shown the invariance of the magnetic moment μ along a particle trajectory if the magnetic field is slowly varying in space. Let's show that it is also invariant if $\mathbf{B}$ is homogeneous but slowly varying in time.

If $\partial_t \mathbf{B} \neq 0$, there will be an electric field associated to this time variation, and a change of the kinetic energy $\mathscr{E}_\perp$ of the particle due to the work of this electric force along its cyclotron trajectory. Its small variation during a cyclotron cycle is

$$\delta\mathscr{E}_\perp = \oint q\mathbf{E}\cdot\mathbf{v}_\perp dt = -|q|\oint \mathbf{E}\cdot d\boldsymbol{\ell} = |q|\iint \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \tag{5.49}$$

where we used the Stokes theorem and Faraday's law to get the last part of the equality, and oriented $d\ell$ in the anti-clockwise direction, consistently with the Stokes law.

Now we assume that the magnetic field is *slowly varying*, that is, its variation on a gyroperiod is a small quantity δB. Then to a good approximation, one has

$$\delta\mathscr{E}_\perp = |q|\frac{\delta B}{\delta t}\iint dS = \frac{q\omega_c \delta B}{2\pi}\pi\rho^2 = \mathscr{E}_\perp\frac{\delta B}{B} \tag{5.50}$$

from which we finally obtain that the small variation

$$\delta\left(\frac{\mathscr{E}_\perp}{B}\right) = \delta\mu = 0. \tag{5.51}$$

The magnetic moment of the particle is, as a consequence, approximately invariant for slow variations of the magnetic field.

5.4.3 A curved field line in free space

We have investigated the effect of curvature of the magnetic field lines. But curvature alone of the field lines is not permitted by Maxwell's equation : a gradient of the field lines must accompany this curvature ; as a consequence, a grad-B drift always accompanies a curvature drift.

1. Consider a curved field line with curvature radius R_c. Calculate the perpendicular gradient $\boldsymbol{\nabla} B$ of field at a point on this field line.

2. Deduce the expression of the perpendicular drift of a particle moving along this field line.

Chapter 6: Kinetic theory

In this chapter, we introduce the most fundamental description of a plasma : the kinetic description, which aims to describes, etymologically speaking, the motion (κινητικός) of the particles of the plasma. Since there are a lots of them, the kinetic treatment will involve an important reduction of the amount of information available, usually down to a single particle phase space distribution function. We will first derive the equations which describe the time-evolution of this phase space distribution, then see how this description connects to the fluid description of plasmas. Finally we investigate the linear behaviour of an electron plasma in the kinetic description, and introduce the important notion of Landau damping of plasma oscillations.

6.1 Time-evolution of the phase space distribution function

We assume that a particle of the specie s, labelled p, evolves in phase space according to the laws of motion

$$\frac{d\mathbf{r}_p}{dt} = \mathbf{v}_p \tag{6.1}$$

$$\frac{d\mathbf{v}_p}{dt} = \frac{\mathbf{F}(\mathbf{r}_p, \mathbf{v}_p)}{m_s} \tag{6.2}$$

and we make the supplementary assumption that the force field is divergent-free in velocity space [1], that is

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{F}(\mathbf{r}, \mathbf{v}) = 0. \tag{6.3}$$

1. You may check that this is valid for the Lorentz force, using $\nabla \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\nabla \times \mathbf{x}) - \mathbf{x} \cdot (\nabla \times \mathbf{y})$

6.1.1 Microscopic dynamics, Klimontovitch's equation

We introduce the distribution of N_s point particles of specie s in a 6 dimensional phase space [2],

$$\tilde{f}_s(\mathbf{r}, \mathbf{v}, t) = \sum_{p=1}^{N_s} \delta(\mathbf{v} - \mathbf{v}_p(t))\delta(\mathbf{r} - \mathbf{r}_p(t)) \tag{6.4}$$

which is sometimes called the Klimontovitch distribution function. This function contains all the information there is to know about the system of point particles, and its evolution according to the laws of physics is completely deterministic (no loss of information) and given by eqs.(6.2). The integral of this distribution on a small phase space volume $d^3\mathbf{v}d^3\mathbf{r}$ gives the number of particles in this volume, and the integral of the distribution on velocity space gives the number density of particles of specie s,

$$\tilde{n}_s(\mathbf{r}, t) = \int \tilde{f}_s(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{v} = \sum_{p=1}^{N_s} \delta(\mathbf{r} - \mathbf{r}_p(t)). \tag{6.5}$$

This density (and the associated current) can be used in principle to calculate the fields $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ and to close the microscopic system of equations. The number of particles in a plasma (say, of the order of the Avogadro number $N_A \sim 10^{23}$), however, makes it in practice impossible to use. This justifies the introduction of a smoothed distribution function, which includes a reduced information on the system and will be more convenient to use in practice.

First, we look for an evolution equation for the distribution $\tilde{f}_s(\mathbf{r}, \mathbf{v}, t)$. For this we compute its time derivative (sommation on the index p is assumed in all the following) :

$$\frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} = -\mathbf{v}\delta(\mathbf{v} - \mathbf{v}_p)\frac{\partial}{\partial \mathbf{r}}\delta(\mathbf{r} - \mathbf{r}_p) - \frac{\mathbf{F}(\mathbf{r}, \mathbf{v})}{m_s}\delta(\mathbf{r} - \mathbf{r}_p)\frac{\partial}{\partial \mathbf{v}}\delta(\mathbf{v} - \mathbf{v}_p) \tag{6.6}$$

where we expanded the derivative of the product of two functions, and used the property of the delta function $\mathbf{x}\delta(\mathbf{x} - \mathbf{y}) = \mathbf{y}\delta(\mathbf{x} - \mathbf{y})$.

$$\frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} = -\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\delta(\mathbf{r} - \mathbf{r}_p)\delta(\mathbf{v} - \mathbf{v}_p) - \frac{\mathbf{F}(\mathbf{r}, \mathbf{v})}{m_s}\frac{\partial}{\partial \mathbf{v}}\delta(\mathbf{v} - \mathbf{v}_p)\delta(\mathbf{r} - \mathbf{r}_p) \tag{6.7}$$

so that

$$\frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{r}} + \frac{\mathbf{F}(\mathbf{r}, \mathbf{v})}{m_s} \cdot \frac{\partial \tilde{f}_s(\mathbf{r}, \mathbf{v}, t)}{\partial \mathbf{v}} = 0 \tag{6.8}$$

which gives the time evolution of the distribution $\tilde{f}_s$ and is usually named Klimontovitch's equation. This is a bit formal, but has the advantage of encompassing all the dynamics of the N_s particles into a single equation.

2. The $\tilde{x}$ (tilde) symbol is used to denote the microscopic, fastly variable, quantities, by opposition to the smoothed quantities that will be introduced in the following of the section

If the force field is divergence free in velocity space (as hypothesized in the previous section) this equation can be cast in the conservative form

$$\frac{\partial \tilde{f}_s(\mathbf{r},\mathbf{v},t)}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{v}\tilde{f}_s(\mathbf{r},\mathbf{v},t) + \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\mathbf{F}(\mathbf{r},\mathbf{v})}{m_s}\tilde{f}_s(\mathbf{r},\mathbf{v},t) = 0, \tag{6.9}$$

or

$$\frac{\partial \tilde{f}_s(\mathbf{r},\mathbf{v},t)}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{U}\tilde{f}_s(\mathbf{r},\mathbf{v},t) = 0, \tag{6.10}$$

which shows that Klimontovitch's equation is a conservation equation in phase space, associated to a flow of 6-velocity $\mathbf{U} \equiv (\mathbf{v}, \mathbf{F}/m_s)$.

6.1.2 Mean-field dynamics, the kinetic equations

In the fashion of fluid dynamics, we introduce a mesoscopic volume in phase space $d^3\mathbf{v}d^3\mathbf{r}$, containing enough particles for averages to make sense, and average the distribution over it [3]. The averaging procedure can be thought of as counting the number of particles ΔN_s in a phase space volume $\Delta V_v \Delta V_r$, dividing by the result by the phase space volume to get a smoothed distribution $f(\mathbf{r},\mathbf{v},t)$,

$$f_s(\mathbf{r},\mathbf{v},t) \simeq \frac{\int_{\Delta V_r, \Delta V_v} \tilde{f}_s(\mathbf{r},\mathbf{v},t) d^3\mathbf{v}d^3\mathbf{r}}{\Delta V_r \Delta V_v} \tag{6.11}$$

This average makes sense only if the r.m.s fluctuations δf_s are small compared to the expectation value $f_s(\mathbf{r},\mathbf{v},t)$. The ratio $\delta f_s/f_s \sim 1/\sqrt{\Delta N_s}$, so one must have a large number of particles in our mesoscopic phase space volumes. Given, as a rough order of magnitude [4], $f_s \sim (\ell v_{th})^{-3}$, the phase space volume must be such that $\Delta V_v \Delta V_r \gg (\ell v_{th})^3 = v_{th}^3/n$. On the other hand, we need this phase space volume to be small enough to resolve the important scales of the plasma. Typically, the Debye length in real space, and the thermal speed in velocity space. It is possible to construct such a volume, because the plasma parameter $\Gamma = (n\lambda_D)^{-3} \ll 1$. Indeed, taking $\Delta r = \alpha\lambda_D$ with $\alpha < 1$ (so that we can resolve the Debye scale, for instance consider $\alpha = 0.1$). We have $\Delta r = \alpha\ell\Gamma^{-1/3}$. We must take $\Delta v \gg \ell v_T \ /\Delta r = (v_T/\alpha)\Gamma^{1/3}$ to have a large number of particle in the mesoscopic volume, while keeping Δv small compared to the thermal speed. This is possible because of the smallness of Γ. Typically we can chose $\alpha \sim \Gamma^{1/6}$, so that $\Delta r = \Gamma^{1/6}\lambda_D \ll \lambda_D$, $\Delta r = \Gamma^{-1/6}\ell \gg \ell$, which lets space for $\Delta v \gg v_T\Gamma^{1/6}$ while keeping a good "velocity resolution" [5].

3. Formally, this average is an ensemble average, which means that the distribution is averaged over an infinite number of microscopic realizations of the plasma that keep the macroscopic parameters of the plasma the same.

4. ℓ is the interparticular length and v_{th} the thermal speed, as defined in the chapter 2 – plasma scales.

5. In a plasma, $\Gamma^{1/6} \sim 1/50$, typically.

Now that we have constructed our phase-space averaging scheme, we introduce the bracket notation to denote the averaging over phase space mesoscopic volumes,

$$f_s(\mathbf{r}, \mathbf{v}, t) \equiv \langle \tilde{f}_s(\mathbf{r}, \mathbf{v}, t) \rangle. \tag{6.12}$$

We define the smooth field, or *mean-fields* as $\mathbf{E} = \langle \tilde{\mathbf{E}} \rangle$, and $\mathbf{B} = \langle \tilde{\mathbf{B}} \rangle$. We now introduce the fluctuations (which necessarily vanish when the number of particles in a mesoscopic volume tends to infinity, since we have seen they scale as $1/\sqrt{N_s}$),

$$\delta f_s(\mathbf{r}, \mathbf{v}, t) = \tilde{f}_s(\mathbf{r}, \mathbf{v}, t) - f_s(\mathbf{r}, \mathbf{v}, t) \tag{6.13}$$

$$\delta \mathbf{E}(\mathbf{r}) = \tilde{\mathbf{E}}(\mathbf{r}) - \mathbf{E}(\mathbf{r}) \tag{6.14}$$

$$\delta \mathbf{B}(\mathbf{r}) = \tilde{\mathbf{B}}(\mathbf{r}) - \mathbf{B}(\mathbf{r}) \tag{6.15}$$

By construction, the average of the microscopic fluctuation is equal to zero. Using these definitions, we can write the ensemble averaged form of Kilmontovitch's equation as

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = -\frac{q_s}{m_s} \langle (\delta \mathbf{E} + \mathbf{v} \times \delta \mathbf{B}) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \rangle \tag{6.16}$$

The left hand of this equation describes the evolution of the smooth distribution function in phase space, under the action of the smoothed fields. This smooth distribution is the physical tool that is the most adpated for lots of plasma applications involving small scales, but mean-field, or collective, effects (i.e. not collisions, for instance). The right hand term describes the effect of the microscopic field fluctuations on the distribution function. It is the *collision integral*,

$$\mathscr{C}(f_s) = -\frac{q_s}{m_s} \langle (\delta \mathbf{E} + \mathbf{v} \times \delta \mathbf{B}) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \rangle, \tag{6.17}$$

and is in general very complicated to calculate. There are several models employed to replace it, in particular diffusion terms, as we saw in the section on collisions.

In the case where this collision integral vanishes, or is negligible, the kinetic equation (6.16) is called the Vlasov equation, which has a very important role in the modeling of lots of effects in space and astrophysical plasmas ; its used is justified when considering phenomena on timescales much smaller than the collision frequency of the specie s.

6.1.3 Some properties of the kinetic equation

As we did for Klimontovitch's equation, we note that if the force field is divergent free in $\mathbf{v}$-space, then eq.(6.16) is a continuity equation, expressing the conservation of the particle number in phase space,

$$\frac{\partial f_s}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{U} f_s) = \mathscr{C}(f_s), \tag{6.18}$$

where $\mathbf{U} = (\mathbf{v}, \mathbf{F}/m_s)$ is the 6-dimensional velocity vector of the particle's flow in phase space. The collision term expresses the deviation from this conservative flow in phase space. In the absence of collision, Vlasov equation, the flow is perfectly conservative, and can be written as a total derivative

$$\frac{df_s}{dt} = 0. \tag{6.19}$$

We can express the Vlasov equation for any set of variables $\mathbf{x}$ one may want to parametrize the phase space with, as

$$\frac{\partial f_s}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f_s}{\partial \mathbf{x}} = 0. \tag{6.20}$$

From which we see that a distribution function that depends only on the constants of the motion of a particle is always a steady-state solution of the Vlasov equation :

$$\frac{\partial f_s(C_i)}{\partial t} = 0. \tag{6.21}$$

This is a convenient way to find equilibrium solutions to the Vlasov equation. For instance, a distribution function that would depend only on the particle's energy will always be a steady state solution of the Vlasov equation.

Eq.(6.20) is also convenient to find simplified kinetic equation benefitting from known invariants of the motion. For instance, a charged particle in a magnetic field has its magnetic moment conserved (for slow variations of the magnetic field) ; in the absence of an electric field, it also has its total kinetic energy conserved (or modulus of the velocity vector). Therefore, the gyrophased averaged distribution function will follow the simple equation

$$\frac{\partial f_s(v^2, \mu, \mathbf{R}_g)}{\partial t} + \mathbf{V}_g \cdot \frac{\partial f_s(v^2, \mu, \mathbf{R}_g)}{\partial \mathbf{R}_g} = 0. \tag{6.22}$$

in which v and μ are just conserved quantities. This kind of equation which makes it possible to easily calculate the kinetic evolution of a population in a given magnetic field configuration is an example of a drift-kinetic equation.

6.2 Link to the fluid equations

The fluid variable (n, $\mathbf{u}$, $\mathbf{p}$ etc.) are defined as the statistical moments with respect to velocity of the phase space distribution function $f(\mathbf{r}, \mathbf{v}, t)$ (we forget the subscript s, but it is implicit ; there is of course a set of statistical moments attributed to each specie). We define the average of ψ over the distribution function as

$$\langle \psi \rangle (\mathbf{r}, t) = \frac{1}{n(\mathbf{r}, t)} \int d^3\mathbf{v} f(\mathbf{r}, \mathbf{v}, t) \psi(\mathbf{r}, \mathbf{v}, t), \tag{6.23}$$

where the normalization factor is the number density

$$n(\mathbf{r}, t) = \int d^3\mathbf{v} f(\mathbf{r}, \mathbf{v}, t). \tag{6.24}$$

We have just seen that the kinetic equation generally reads, where the collision integral $\mathscr{C}$ can be attributed different functional forms, according to the model chosen.

$$\frac{\partial f(\mathbf{r},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f(\mathbf{r},\mathbf{v},t)}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial f(\mathbf{r},\mathbf{v},t)}{\partial \mathbf{v}} = \mathscr{C}(f) \tag{6.25}$$

6.2.1 Density

The number density is defined by eq.(6.24). Its evolution is obtained by integrating the kinetic equation (6.25) over velocity. We obtain

$$\frac{\partial n}{\partial t} + \nabla\cdot n\mathbf{u} = 0 \tag{6.26}$$

where the distribution function has been assumed to vanish at infinity, and we defined the mean velocity as $\mathbf{u} = \langle\mathbf{v}\rangle$. The property of the collision operator $\int d^3\mathbf{v}\mathscr{C}(f) = 0$ has been used – which means that there is no creation, or anihilation, of particles of specie s during a collision (for instance, no chemical or nuclear reaction).

6.2.2 Momentum

We look for the evolution of the momentum density, by multiplying the kinetic equation (6.25) by $m\mathbf{v}$ and integrating over velocity. We obtain

$$\frac{\partial mn\mathbf{u}}{\partial t} + \nabla\cdot\mathbf{\Pi} = n\mathbf{F} + n\mathbf{R}. \tag{6.27}$$

The full stress tensor is defined as $\mathbf{\Pi} = mn\langle\mathbf{vv}\rangle$ and the friction force $\mathbf{R}$ such as

$$n\mathbf{R} = \int d^3\mathbf{v}m\mathbf{v}\mathscr{C}(f). \tag{6.28}$$

This term describes the exchange of momentum between the population of particles described by the distribution function $f(\mathbf{r},\mathbf{v},t)$, and external systems through the term $\mathscr{C}(f)$. These external systems can be other populations of particles, or for instance fields fluctuations, described in a statistical manner.

The stress tensor can be separated into internal stresses and 'external' stresses, by introducing the random (or centered) velocity component $\mathbf{w} = \mathbf{v} - \mathbf{u}$. Then

$$\mathbf{\Pi} = mn\mathbf{uu} + mn\langle\mathbf{ww}\rangle = mn\mathbf{uu} + p\mathbf{I} + \boldsymbol{\pi} \tag{6.29}$$

where the internal stress tensor $mn\langle\mathbf{ww}\rangle$ has been separated between its isotropic part (the pressure tensor) and the, properly said, internal stresses $\boldsymbol{\pi}$ accounting for all the departures from isotropy.

$$\mathbf{\Pi} = mn\mathbf{uu} + \mathbf{p} \tag{6.30}$$

$$\mathbf{p} = mn \langle (\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u}\rangle \tag{6.31}$$

The pressure $\mathbf{p}$ accounts for the tendency of the particle's population to expand under the action of its (centered) thermal motion.

Noting that $\nabla \cdot \mathbf{uu} = \mathbf{u} \cdot \nabla \mathbf{u} + (\nabla \cdot \mathbf{u})\mathbf{u}$, and using the continuity equation (6.26) one can re-write the equation (6.27) as

$$nm \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p - \nabla \cdot \boldsymbol{\pi} + n\mathbf{F} + n\mathbf{R} \tag{6.32}$$

$$nm \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla \cdot \mathbf{p} + n\mathbf{F} + n\mathbf{R} \tag{6.33}$$

which is the form under which is usually presented the Navier-Stokes equation.

6.2.3 Energy

We obtain the equation describing energy conservation by multiplying the kinetic equation by $\frac{1}{2}mv^2$ and integrating over velocity space. One obtains

$$\frac{\partial}{\partial t} \left(\frac{1}{2} nmu^2 + \frac{3}{2} nkT \right) + \nabla \cdot \boldsymbol{\Phi} = n\mathbf{u} \cdot \mathbf{F} + n\mathbf{u} \cdot \mathbf{R} + nQ \tag{6.34}$$

where the kinetic temperature is defined such as $m\mathbf{w}^2 = 3kT$. The particle energy flux density $\boldsymbol{\Phi}$ is defined by $\boldsymbol{\Phi} = n \left\langle \frac{1}{2}mv^2\mathbf{v} \right\rangle$. Q is the heat exchanged per particle and per unit time between the particle system and external macroscopic systems, it is defined by

$$nQ = \int d^3\mathbf{v} \frac{1}{2} mw^2 \mathscr{C}(f), \tag{6.35}$$

and obviously

$$\int d^3\mathbf{v} \frac{1}{2} mv^2 \mathscr{C}(f) = nQ + n\mathbf{u} \cdot \mathbf{R}. \tag{6.36}$$

The energy flux density can be separated into internal and convective terms,

$$\boldsymbol{\Phi} = n \left\langle \frac{1}{2} mw^2 \mathbf{v} \right\rangle + n \left\langle \frac{1}{2} mu^2 \mathbf{v} \right\rangle + n \left\langle m(\mathbf{w} \cdot \mathbf{u})\mathbf{u} \right\rangle + n \left\langle m(\mathbf{w} \cdot \mathbf{u})\mathbf{w} \right\rangle, \tag{6.37}$$

so that

$$\boldsymbol{\Phi} = \left(\frac{1}{2} mu^2 + \frac{3}{2} kT \right) n\mathbf{u} + (\mathbf{p} + \boldsymbol{\pi}) \cdot \mathbf{u} + \mathbf{q} \tag{6.38}$$

where the heat flux density is defined as $\mathbf{q} = n \left\langle \frac{1}{2} mw^2 \mathbf{w} \right\rangle$.

It is possible to obtain an equation that describes only the equation of the internal energy of the gas by subtracting the work part. This latter can be obtained by taking the scalar product of the momentum equation (6.27) by $\mathbf{u}$. Noting that for any scalar quantity ψ one has the equality

$$n\frac{d\psi}{dt} \equiv n\left(\frac{\partial\psi}{\partial t} + \mathbf{u}\cdot\nabla\psi\right) = \frac{\partial}{\partial t}n\psi + \nabla\cdot\psi\mathbf{u}, \tag{6.39}$$

(where we used the continuity equation $dn/dt = -n\nabla\cdot\mathbf{u}$), so that for instance

$$nm\mathbf{u}\cdot\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \frac{\partial}{\partial t}\frac{nmu^2}{2} + \nabla\cdot\frac{nmu^2}{2}\mathbf{u} \tag{6.40}$$

we obtain

$$\frac{\partial}{\partial t}\frac{nmu^2}{2} + \nabla\cdot\frac{nmu^2}{2}\mathbf{u} = \mathbf{u}\cdot(-\nabla p - \nabla\cdot\boldsymbol{\pi} + n\mathbf{F} + n\mathbf{R}) \tag{6.41}$$

which expresses the conservation of the macroscopic kinetic energy. Subtracting this equation from the total energy equation (6.34), we obtain

$$\frac{\partial}{\partial t}\frac{3nkT}{2} - \nabla\cdot\frac{nmu^2}{2}\mathbf{u} + \nabla\cdot\boldsymbol{\Phi} = \mathbf{u}\cdot(\nabla p + \nabla\cdot\boldsymbol{\pi}) + nQ. \tag{6.42}$$

Now using the equation (6.38) for the energy flux density, and simplifying the expression using again the equality (6.39), we finally obtain

$$n\frac{d}{dt}\frac{3kT}{2} = -\nabla\cdot\mathbf{q} - (\mathbf{p}+\boldsymbol{\pi}) : \nabla\mathbf{u} + nQ, \tag{6.43}$$

where $\mathbf{a} : \nabla\mathbf{b} = a_{ij}\partial_{x_j}b_i$, so that if $\mathbf{p} = p\mathbf{I}$, $\mathbf{p} : \nabla\mathbf{u} = p(\nabla\cdot\mathbf{u})$.

6.2.4 Entropy

The internal energy equation can be expressed, equivalently, as an equation describing the entropy production. For this we introduce the entropy per particle of a gas of non-interacting point particles in density-temperature representation,

$$S(n,T) = k\left[\ln\left(\frac{1}{n}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right) + \frac{5}{2}\right], \tag{6.44}$$

or, since the expression of the constant does not matter here,

$$S(n,T) = k\ln\frac{T^{3/2}}{n} + const. \tag{6.45}$$

Therefore the time evolution of the entropy is given by (using eq.6.43)

$$\frac{dS}{dt} + \frac{k}{n}\frac{dn}{dt} = -\frac{\nabla\cdot\mathbf{q}}{nT} - \frac{(\mathbf{p}+\boldsymbol{\pi}) : \nabla\mathbf{u}}{nT} + \frac{Q}{T} \tag{6.46}$$

then using the form $\mathbf{p} = (nkT)\mathbf{I}$, one obtains

$$\frac{dS}{dt} = -\frac{\nabla\cdot\mathbf{q}}{nT} - \frac{\boldsymbol{\pi} : \nabla\mathbf{u}}{nT} + \frac{Q}{T}. \tag{6.47}$$

So that the internal energy equation states that the increase of entropy is due to three kind of irreversible processes : heat conduction, viscosity (both internal to the gas) and exchange of heat with external systems.

6.3 The Vlasov-Poisson system in the linear approximation : kinetic plasma oscillations

We consider an ion-electron plasma, unmagnetized and perfectly collisionless. Each plasma population is described with the Vlasov equation

$$\frac{\partial f_s(\mathbf{r},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f_s(\mathbf{r},\mathbf{v},t)}{\partial \mathbf{r}} + \frac{q_s\mathbf{E}}{m_s}\cdot\frac{\partial f_s(\mathbf{r},\mathbf{v},t)}{\partial \mathbf{v}} = 0 \tag{6.48}$$

6.3.1 The kinetic plasma parallel dielectric function

We want to find the kinetic dispersion relation of plasma waves. For this we assume the existence of a small electric field $\mathbf{E}_1(\mathbf{r},t)$ in the plasma, associated to a small perturbations $f_{s,1}(\mathbf{r},\mathbf{v},t)$ of the distribution function, which can then be written $f_s(\mathbf{r},\mathbf{v},t) = f_{s,0}(\mathbf{v}) + f_{s,1}(\mathbf{r},\mathbf{v},t)$. We look for the expression of $f_{s,1}$ as a function of $\mathbf{E}_1$.

The linearized Vlasov equation for the specie s is

$$\frac{\partial f_{s,1}(\mathbf{r},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{s,1}(\mathbf{r},\mathbf{v},t)}{\partial \mathbf{r}} + \frac{q_s}{m_s}\mathbf{E}_1\cdot\frac{\partial f_{s,0}(v)}{\partial \mathbf{v}} = 0. \tag{6.49}$$

The Maxwell-Gauss (or Poisson) equation gives us the equation for the electric field

$$\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{E}_1(\mathbf{r},t) = \frac{1}{\varepsilon_0}\sum_s q_s n_{s,1}(\mathbf{r},t) = \frac{1}{\varepsilon_0}\sum_s q_s\int f_{s,1}(\mathbf{r},\mathbf{v},t)d^3\mathbf{v} \tag{6.50}$$

where we assumed that the unperturbed plasma is quasi neutral, so that $\sum_s q_s n_{s,0} = 0$. We write the electric field and distribution function perturbation as continuous Fourier series (in space and time),

$$f_{s,1}(\mathbf{r},v,t) = \int\int f_{s,1k\omega}(v)e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}d^3\mathbf{k}d\omega \tag{6.51}$$

$$f_{s,1k\omega}(\mathbf{v}) = \frac{1}{(2\pi)^4}\int\int f_{s,1}(\mathbf{r},\mathbf{v},t)e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}d^3\mathbf{r}dt \tag{6.52}$$

And for the electric field

$$\mathbf{E}_1(\mathbf{r},t) = \int\int \mathbf{E}_{1k\omega}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}d^3\mathbf{k}d\omega \tag{6.53}$$

$$\mathbf{E}_{1k\omega} = \frac{1}{(2\pi)^4}\int\int \mathbf{E}_1(\mathbf{r},\mathbf{v},t)e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}d^3\mathbf{r}dt \tag{6.54}$$

The Vlasov equation can be written for the Fourier components as

$$i(-\omega + \mathbf{k}\cdot\mathbf{v})f_{s,1k\omega} + \frac{q_s\mathbf{E}_{1k\omega}}{m_s}\cdot\frac{\partial f_{s,0}(v)}{\partial \mathbf{v}} = 0 \tag{6.55}$$

so that the perturbation of the distribution function is [6]

$$f_{s,1k\omega} = -\frac{iq_s\mathbf{E}_{1k\omega}}{m_s(\omega - \mathbf{k}\cdot\mathbf{v})}\cdot\frac{\partial f_{s,0}}{\partial \mathbf{v}} \tag{6.56}$$

We now express the Fourier transformed Maxwell-Gauss equation

$$i\mathbf{k}\cdot\mathbf{E}_{1k\omega} = \frac{1}{\varepsilon_0}\sum_s q_s \int f_{s,1k\omega}(\mathbf{v})d^3\mathbf{v}, \tag{6.57}$$

and insert it in eq.(6.56). We get

$$\mathbf{k}\cdot\mathbf{E}_{1k\omega} = -\sum_s \frac{q_s^2}{m_s\varepsilon_0}\int \frac{\mathbf{E}_{1k\omega}\cdot\nabla_v f_{s,0}}{\omega - \mathbf{k}\cdot\mathbf{v}}d^3\mathbf{v} \tag{6.58}$$

This equation informs us on the component of the electric field parallel to the wavevector $\mathbf{k}$. We introduce the notation $\|$ for the direction parallel to $\mathbf{k}$, and $\perp$ for the component perpendicular to it. We have

$$kE^{\|}_{1k\omega}\left(1 + \sum_s \frac{q_s^2}{m_s\varepsilon_0 k}\int \frac{\nabla_{v_\|} f_{s,0}(\mathbf{v})}{\omega - kv_\|}d\mathbf{v}_\perp dv_\|\right) = 0 \tag{6.59}$$

The term in the parenthesis must be equal to zero for the Fourier components of the electric field to be non-vanishing. This term is the longitudinal dielectric function $\varepsilon^{\|}(\omega,k)$ of the plasma [7], which can be decomposed in terms of the longitudinal susceptibilities [8] of the different species $\chi^{\|}_s(\omega,k)$,

$$\varepsilon^{\|}(\omega,k) = 1 + \sum_s \frac{q_s^2}{m_s\varepsilon_0 k}\int \frac{\nabla_{v_\|} f_{s,0}(\mathbf{v})}{\omega - kv_\|}d\mathbf{v}_\perp dv_\| = 1 + \sum_s \chi^{\|}_s(\omega,k). \tag{6.60}$$

It is convenient to introduce the reduced (and normalized) distribution function $\bar{f}_{s,0}(v_\|)$,

$$\bar{f}_{s,0}(v_\|) = \frac{1}{n_0}\int f_{s,0}(\mathbf{v})d\mathbf{v}_\perp, \tag{6.61}$$

so that $\int \bar{f}_{s,0}(v_\|)dv_\| = 1$. The longitudinal susceptibilities now read

$$\chi^{\|}_s(\omega,k) = \frac{\omega_{p,s}^2}{k}\int \frac{\nabla_{v_\|}\bar{f}_{s,0}(v_\|)}{\omega - kv_\|}dv_\| \tag{6.62}$$

6. Note that we are here dividing by $(-\omega + \mathbf{k}\cdot\mathbf{v})$, which can be nul – and certainly will be, for some values of v – corresponding to resonant particles.

7. One sees directly from eq.(6.57), adding an extra source term ρ_{ext} on the right-hand side, that the Maxwell-Gauss equation reads $ik\varepsilon^{\|}(\omega,k)E^{\|}_{1k\omega} = \rho_{ext}/\varepsilon_0$.

8. $\chi^{\|}_s$ can also be named the polarizability of the specie s. Indeed one easily see that the parallel component of the polarization vector associated to the specie s checks $P^{\|}_{1,s,\omega k} = q_s n_{1,s,\omega k} = \varepsilon_0\chi^{\|}_{s,\omega k}E^{\|}_{1\omega k}$

6.3.2 Electron plasma longitudinal waves

The dispersion relation $\varepsilon^{\parallel}(\omega, k) = 0$ has a regularity problem at $v_{\parallel} = \omega/k$, which is the wave-particle resonance : for this velocity, the electrons have exactly the phase velocity of the (ω, k) electrostatic mode, and, in the linear limit, interact "forever" with this mode, exchanging an infinite amount of energy from it. In the following, we have to assume that, $\omega - kv_{\parallel}$ never vanishes – for this we consider values of the phase speed $v_{\varphi} = \omega/k$ much larger than the thermal spread of the distribution function, so that there are practically no particles having velocities $v_{\parallel} = v_{\varphi}$; the distribution function may be assumed to be equal to zero at this point. From now on, we consider an electron plasma with ions at rest, so $\chi_i^{\parallel}(\omega, k) = 0$. We also drop the "parallel" indices for the parallel distribution functions.

First, we get rid of the partial derivative in the integrand in the expression of the electron susceptibility by integrating by parts

$$\int \frac{\partial_v \bar{f}_{e,0}(v)}{\omega - kv} dv = \left[\frac{\bar{f}_{e,0}(v)}{\omega - kv} \right]_{-\infty}^{\infty} - k \int \frac{\bar{f}_{e,0}(v)}{(\omega - kv)^2} dv \tag{6.63}$$

The term between bracket is nul since the distribution function vanishes at infinity. We can therefore re-express the susceptibility as

$$\chi_e(\omega, k) = -\omega_{p,e}^2 \int \frac{\bar{f}_{e,0}(v)}{(\omega - kv)^2} dv. \tag{6.64}$$

Cold plasma

Neglecting the thermal motion of the electrons ($T_e \to 0$), we can approximate $\bar{f}_{e,0}(v) \simeq \delta(v)$. We then get from eqs.(6.60)-(6.64)

$$1 - \frac{\omega_p^2}{\omega^2} = 0 \quad \Rightarrow \quad \omega = \pm\omega_p \tag{6.65}$$

This is the dispersion relation that we already obtained previously in the course by neglecting the pressure term in the fluid equation.

Warm plasma

In the fluid treatment of the plasma oscillation made in chapter 2, it was not clear how to deal with the pressure term (we needed a closure relation, the choice of which was not easy to justify). The kinetic treatment overcomes this problem, and justifies the fluid closure that should be used. We assume that $\omega/k \gg v$ and Taylor expand the integrand :

$$(\omega - kv)^{-2} = \omega^{-2}(1 - kv/\omega)^{-2} = \omega^{-2}\left(1 + 2kv/\omega + 3(kv/\omega)^2 + ...\right) \tag{6.66}$$

Introducing this into eqs.(6.60)-(6.64), we get

$$1 = \frac{\omega_p^2}{\omega^2} \int \bar{f}_{e,0}(v) \left(1 + 2kv/\omega + 3(kv/\omega)^2 + ...\right) dv \tag{6.67}$$

or, introducing the mean values of the electron velocities and the wave's phase velocity $v_\varphi = \omega/k$,

$$1 = \frac{\omega_p^2}{\omega^2} \left(1 + 2\frac{\langle v \rangle}{v_\varphi} + 3\frac{\langle v^2 \rangle}{v_\varphi^2} + ...\right). \tag{6.68}$$

We must assume that there is no net drift of the electron population with respect to the ion one, so $\langle v \rangle = 0$ (if not our treatment should include the presence of a net current). We get finally the dispersion relation, in the limit $\omega \simeq \omega_p$ (consistent with our assumption that $v_\varphi^2 \gg \langle v^2 \rangle$) [9],

$$\omega^2 \simeq \omega_p^2 + 3k^2 \langle v^2 \rangle \tag{6.69}$$

Taking into account the thermal spread of the distribution function then induces a modification of the dispersion relation. The energy now propagates at a non-zero group velocity,

$$v_g = \frac{\partial \omega}{\partial k} \simeq \frac{3\langle v^2 \rangle}{v_\varphi} \tag{6.70}$$

which is a small quantity with respect to the electron thermal speed, since the phase velocity had to be assumed much larger.

6.3.3 Landau damping and beam instability

We have neglected the effect of the pole (vanishing of the denominator $\omega - kv$) by considering waves fast enough for no particle to resonate with. Actually the integral appearing in the expression of the polarizability is improper around $v = v_\varphi$. Dealing with this problem requires a complicated mathematical treatment, involving integration around carefully chosen contours in the complex plane, that was made by Landau in his paper *The vibration of the electronic plasma*, in 1946. The result of this analysis can be summarized though the so-called Plemelj formula,

$$\lim_{\epsilon \to 0^+} \frac{g(x)}{x - u \pm i\epsilon} \equiv \mp i\pi\delta(x-u) + \mathscr{P}\left(\frac{1}{x-u}\right) \tag{6.71}$$

with $\mathscr{P}$ the Cauchy principal value operator. Applying this to eq.(6.64), we have

$$\chi_s^{\parallel} = \frac{\omega_p^2}{k} \left(\mathscr{P} \int \frac{\partial_v \bar{f}_{e,0}(v)}{\omega - kv} dv - i\pi \int \partial_v \bar{f}_{e,0}(v) \delta(\omega - kv) dv\right) \tag{6.72}$$

The principal part is basically the value of the integral by setting the number of particles around the singularity equal to zero. So in the limit of large phase velocity compared to

9. Instead of this approximation, one can solve exactly the polynome to get $\omega^2 = \omega_p^2/2(1 + \sqrt{1 + 12\langle v^2 \rangle k^2/\omega_p^2})$

the thermal velocity of the particles, we may Taylor expand it and get the same result as in the previous section. The dispersion relation is (where we used $\delta(ax) = \delta(x)/|a|$).

$$1 = \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{\langle v^2\rangle}{v_\varphi^2} + ...\right) + i\pi\text{sign}(k)\frac{\omega_p^2}{k^2}\left.\frac{\partial \bar{f}_{e,0}(v)}{\partial v}\right|_{v=\omega/k} \tag{6.73}$$

Therefore wave frequency ω now has an imaginary part, reflecting exponential growth/damping of the mode (ω, k) with time. Separating the real and imaginary parts as $\omega = \omega_r + i\gamma$, one has

$$\omega_r(k) \simeq \omega_p\left(1 + \frac{3}{2}\frac{\langle v^2\rangle k^2}{\omega_p^2}\right) \tag{6.74}$$

and

$$\gamma(k) \simeq \text{sign}(k)\frac{\pi}{2}\frac{\omega_p^3}{k^2}\left.\frac{\partial \bar{f}_{e,0}(v)}{\partial v}\right|_{v=\omega/k} \tag{6.75}$$

where it has been assumed that $\gamma \ll \omega_p$.

This result is important : electrostatic waves in plasmas constantly exchange energy with the resonant population $v = \omega/k$. This exchange takes the form of a damping (the *Landau damping*) when the slope of the distribution function around the velocity is negative. A Maxwellian plasma will then always damp waves. The rate of this damping can be very small (the damping time very long...) especially for high phase velocity waves. This is the reason why plasma waves are indeed observed in plasmas. But the damping can become very important for waves having smaller phase velocities – in particular of the order of the thermal speed of the electrons, although the treatment provided previously is in this case not exactly correct. Waves with $k \sim \omega_p/v_{th}$ will be heavily damped, and transfer their energy as thermal energy into the plasma. They can be used as heating devices for instance in Tokamaks (RF heating systems).

If the distribution function exhibit a positive slope in some region of v space (an *electron beam*), then $\gamma(k)$ will be positive in some range of wavevectors : plasma waves will grow in this region of k space. This is what is called the *beam-plasma instability.* It explains most of the high amplitude plasma waves observed in the interplanetary medium – and the consequent radio emissions associated with the propagation of electron beams in the interplanetary medium.

6.3.4 Two streams instability

This instability arises when two beams of charged particles (each beam having a thermal spread much smaller that the relative velocity between the beams) interact. We consider the normalized distribution

$$\bar{f}_0(v) = \frac{1}{2}\left(\delta(v - v_0) + \delta(v + v_0)\right). \tag{6.76}$$

The polarisability is in this case

$$\chi_e^{\parallel}(\omega, k) = -\frac{\omega_{p,e}^2}{2}\left(\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}\right). \tag{6.77}$$

so that the dispersion relation to solve for is

$$1 = \frac{\omega_{p,e}^2}{2}\left(\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}\right). \tag{6.78}$$

which is a bi-quadratic equation. It admits imaginary roots if its discriminent is negative, the condition for which is $v_0 > \omega_p/k$: waves will develop with phase speeds smaller than half the beams relative speed. These waves will pump kinetic energy from the beams.

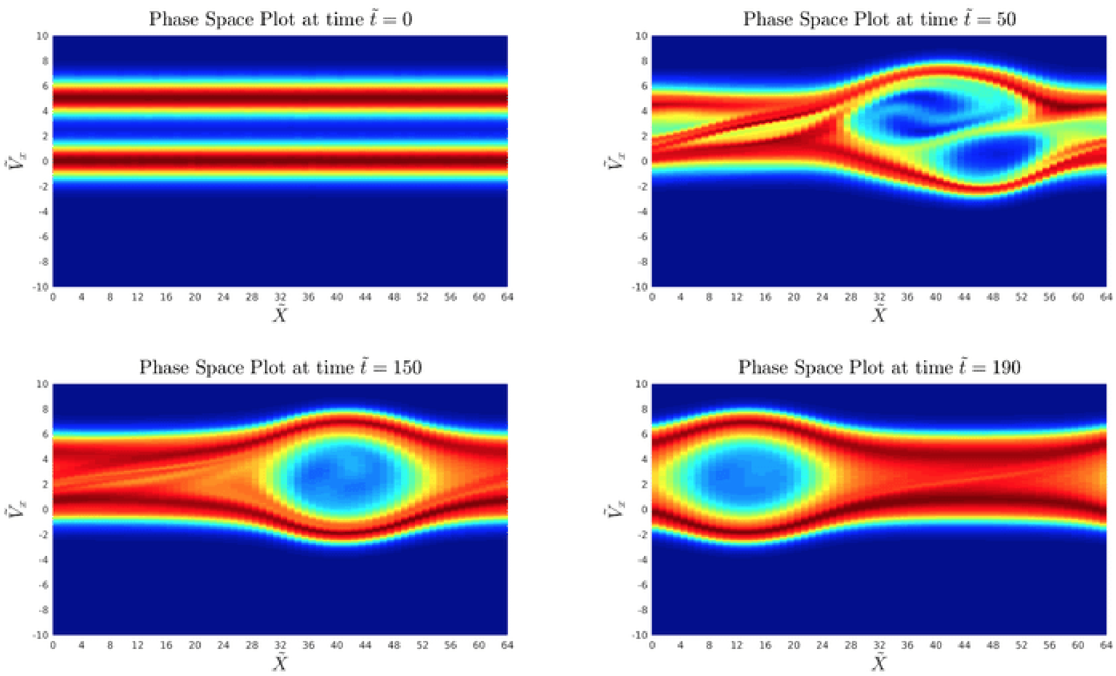


FIGURE 6.1 – Nonlinear evolution of the two-stream instability.

6.4 Examples and exercises

6.4.1 The Harris current sheet

Current sheets are important in plasmas physics, as topological boudaries between magnetic sectors. They are subjected to various kind of instabilities, that lead to magnetic reconnection. To study this processes on the kinetic level, it is important to have a steady-state description of a current sheet in the Vlasov description. Such a model was proposed by Harris in 1962, and is still used to initialize most kinetic numerical simulations.

We consider a Cartesian frame, a current sheet thickness in the x direction and a system invariant by translation along y and z. We take the current density $\mathbf{j}$ such as $\mathbf{j} = j(x)\mathbf{u}_y$, so that $\mathbf{B} = B(x)\mathbf{u}_z$, and the potential vector is $\mathbf{A} = A(x)\mathbf{u}_y$.

To build a stationary solution, we use the property of the Vlasov equation that any distribution function of the dynamic invariants of the particles is a steady state solution. The invariance of the system along y and z implies that the particle's momentum components p_y and p_z are constants. The energy of a particle is also a constant. Our constants read

$$E = \frac{1}{2}m_s(v_x^2 + v_y^2 + v_z^2) + q_s\Phi(x) \tag{6.79}$$

$$p_y = m_s v_y + q_s A_y(x) \equiv m_s\alpha_{2,s} \tag{6.80}$$

$$p_z = m_s v_z \equiv m_s\alpha_{3,s} \tag{6.81}$$

where we introduce the constants α homogeneous to velocities to stick to Harris paper. We now build α_1 from the energy as

$$\alpha_{1,s}^2 = 2E/m_s - \alpha_2^2 - \alpha_3^2 = v_x^2 - \frac{2q_s}{m_s}v_yA_y - \frac{q_s^2}{m_s^2}A_y^2 + 2\frac{q_s}{m_s}\Phi \tag{6.82}$$

in which we eliminated v_y^2 from the equation by using

$$v_y^2 = p_y^2/m_s^2 - q_s^2A_y^2/m_s^2 - 2(q_s/m_s)v_yA_y. \tag{6.83}$$

Then any function $f(\alpha_1, \alpha_2, \alpha_3)$ is a steady-state solution of the Vlasov equation. Harris suggests to look for one in the form

$$f_s(x, \mathbf{v}) = \frac{n_0}{(2\pi v_{th,s})^{3/2}} \exp\left(-\frac{\alpha_{1,s}^2 + (\alpha_{2,s} - u_s)^2 + \alpha_{3,s}^2}{2v_{th,s}^2}\right) \tag{6.84}$$

with $v_{th,s}^3 = kT/m_s$. With a bit of algebra, we check that it can be recast as

$$f_s(x, \mathbf{v}) = \frac{n_0}{(2\pi v_{th,s})^{3/2}} \exp\left(-\frac{q_s(u_sA_y + \Phi)}{kT}\right) \exp\left(-\frac{v_x^2 + (v_y - u_s)^2 + v_z^2}{2v_{th,s}^2}\right) \tag{6.85}$$

We now need to couple this equation to the field. For this we first compute the density and current by integrating on velocity space, we get

$$n_s(x, \mathbf{v}) = n_0 \exp\left(-\frac{q_s(u_sA_y + \Phi)}{kT}\right) \tag{6.86}$$

and

$$\mathbf{j}_s(x, \mathbf{v}) = n_0u_s \exp\left(-\frac{q_s(u_sA_y + \Phi)}{kT}\right)\mathbf{u}_y = n_su_s\mathbf{u}_y \tag{6.87}$$

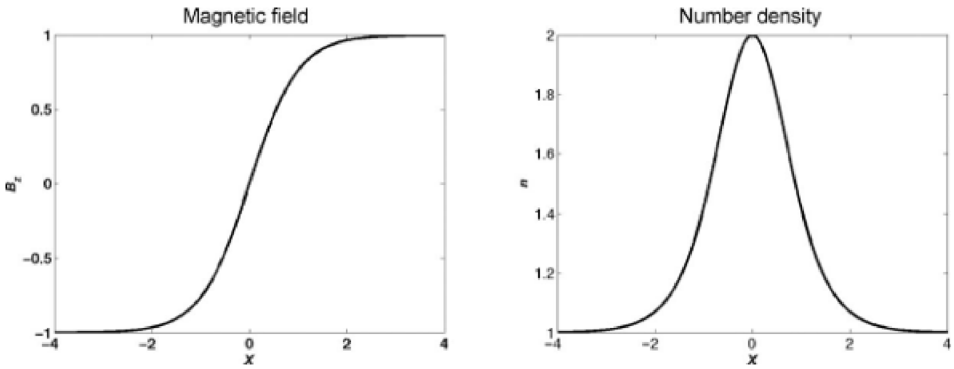


FIGURE 6.2 – Magnetic field and density profile in a Harris current sheet.

The Poisson equation reads

$$\frac{d^2\Phi}{dx^2} = \frac{en_0}{\varepsilon_0}\left(\exp\left(\frac{e(u_eA_y + \Phi)}{kT}\right) - \exp\left(\frac{-e(u_iA_y + \Phi)}{kT}\right)\right) \tag{6.88}$$

we simplify the problem by assuming that $u_e = -u_i = U$ (which is always possible by a proper change of frame), we get

$$\frac{d^2\Phi}{dx^2} = \frac{2en_0}{\varepsilon_0}\exp\left(\frac{eUA_y}{kT}\right)\sinh\frac{e\Phi}{kT} \tag{6.89}$$

The vector potential also satisfies a Poisson equation [10],

$$\frac{d^2A_y}{d^2x} = -\mu_0 j_y, \tag{6.90}$$

from which we get

$$\frac{d^2A_y}{d^2x} = -2\mu_0 e n_0 U \exp\left(\frac{eUA_y}{kT}\right)\cosh\frac{e\Phi}{kT} \tag{6.91}$$

Eq.(6.89) has the trivial solution $\Phi = 0$, which is the only solution for which the plasma is strictly speaking quasi-neutral : we retain this one. We still have to solve eq.(6.91). Multiplying by A'_y on both side and integrating we get

$$\frac{1}{2}A'^2_y = C - 2\mu_0 n_0 kT \exp\left(\frac{eUA_y}{kT}\right) \tag{6.92}$$

The solution to eq.(6.91) is obtained after some complicated calculations

$$A_y(x) = -\frac{2kT}{eU}\ln\cosh\left(\frac{x}{\lambda_D}\frac{U}{c}\right) \tag{6.93}$$

10. From Maxwell-Ampère, using rot(rot) formula and noting that the divergence of A is equal zero from the chosen form $\mathbf{A} = A(x)\mathbf{u}_y$.

where $\lambda_D^2 = \varepsilon_0 kT/ne^2$ as usual. The density of both species is

$$n(x) = n_0 \exp\left(\frac{eUA_y}{kT}\right) = \frac{n_0}{\cosh^2\left(\frac{x}{\lambda_D}\frac{U}{c}\right)}. \tag{6.94}$$