Dynamics of charged particles in a magnetic field

Equation of motion of a particle of charge q and mass m $$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{v} \times \mathbf{B}.$$

First we look at the most simple case : what if the magnetic field is constant (along the z axis) ?

$$\frac{dv_\perp}{dt} + i\frac{qB}{m}v_\perp = 0, \qquad \Rightarrow \qquad v_\perp(t) = v_\perp(0)\exp\left(-i\omega_c t\right)$$

Where ω_c is the cyclotron frequency (angular frequency, to be precise). Can be positive or negative depending on the particle's charge : positive ions rotate clockwise, and electrons anti-clockwise.

The trajectory in the perpendicular plane is given by $$\bar{r}_\perp = \bar{r}_\perp(0) + \frac{i\bar{v}_\perp(0)}{\omega_c}\left(e^{-i\omega_c t} - 1\right).$$

Where we can introduce the Larmor radius $\rho = |\frac{v_\perp}{\omega_c}|$, which is the radius of the perpendicular trajectory.

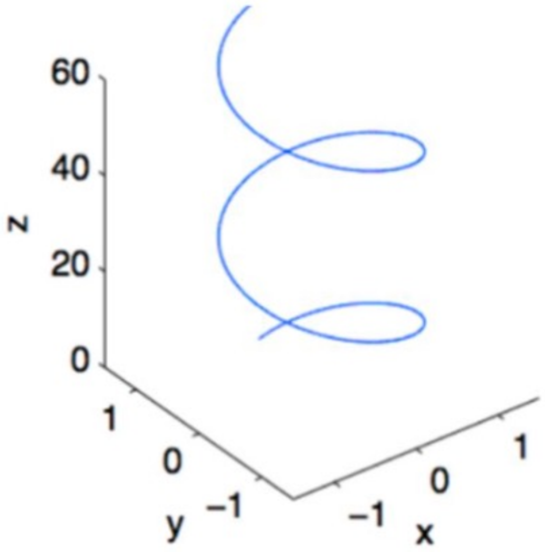
Dynamics of charged particles in a magnetic field

Important scales/parameters for the motion of a particle in a magnetic field:

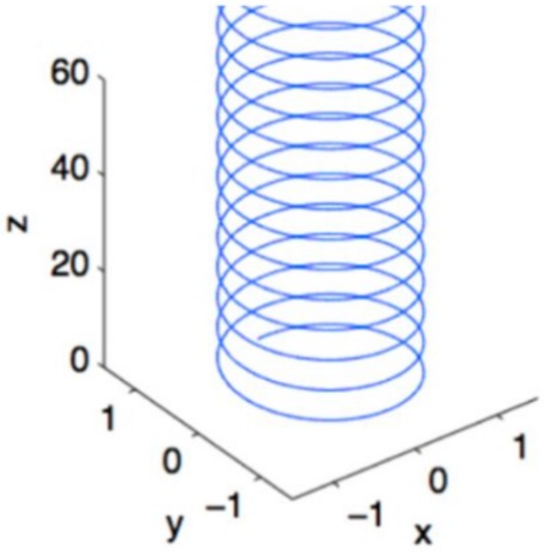
- Cyclotron frequency : $\omega_c = qB/\gamma m$ (careful : it is a signed quantity)
- Larmor radius : $\rho_\ell = v_\perp/|\omega_c|$
- Pitch-angle : $\cos\theta = v_\parallel/v$
- Gyrophase : $\phi(t) = \phi_0 + \omega_c t$
- Guiding center position : $\boldsymbol{R}_g(t) = \boldsymbol{r}(t) - \boldsymbol{r}_\ell(t)$

Pitch-angle

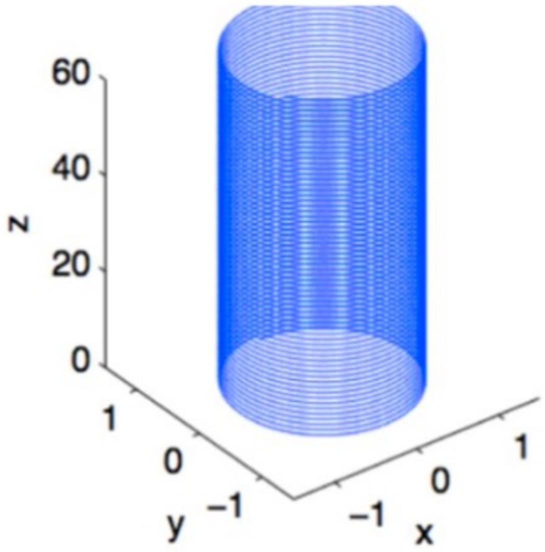
$\theta = 10°$



$\theta = 45°$



$\theta = 80°$



The motion parallel to the magnetic field is $v_z = const.$ and $z(t) = v_z t + z_0$

Therefore the motion of the particle is an helix, with a pitch $p = 2\pi v_z / \omega_c$

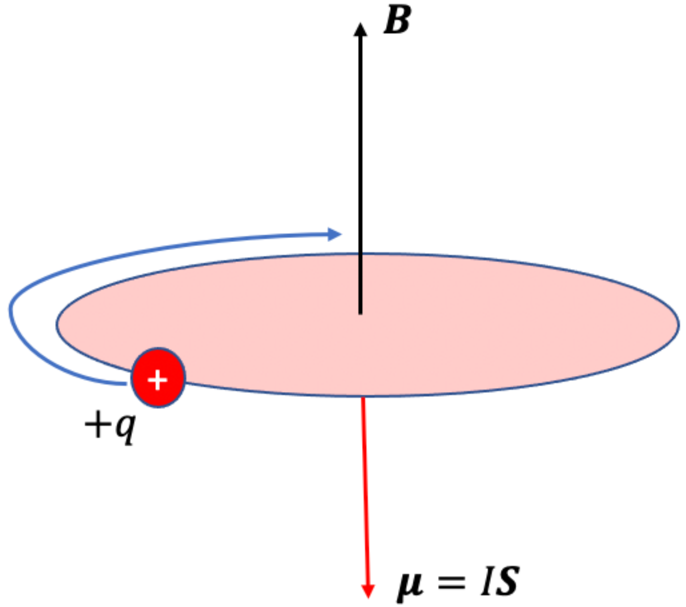
We generally introduce the pitch-angle of the particle such that $\tan\theta = \left|\frac{v_\perp}{v_z}\right| = \frac{2\pi\rho}{p}$

Diamagnetic behaviour of a plasma

One can remark that the current produced by the current loop of a particle rotating in the magnetic field **B** produces a small magnetic field that opposes **B.**

Modeling the particle as a small current loop of radius ρ, one can associate to it a magnetic moment

$$\boldsymbol{\mu} = I\mathbf{S} = -\frac{|q\omega_c|\rho^2}{2}\mathbf{u}_z = -\frac{\mathscr{E}_\perp}{B}\mathbf{b}$$



Diamagnetic behaviour of a plasma

Reminding the Ampère's law in a magnetized medium :

$$\nabla\times\mathbf{B} = \mu_0(j_{ext} + \nabla\times\mathbf{M}) = \nabla\times(\boldsymbol{B_0} + \mu_0\boldsymbol{M})$$

With the medium magnetization vector $\mathbf{M} = n\boldsymbol{\mu}$

So the magnetic field B in the plasma checks

$$\mathbf{B} = \mathbf{B}_0 + \mu_0\mathbf{M} \simeq \left(1 - \frac{nkT}{B_0^2/2\mu_0}\right)\mathbf{B}_0 \simeq (1-\beta)\,\mathbf{B}_0$$

Where we summed the contribution on the electrons and ions in the calculation of the magnetization, assumed ions and electrons of equal temperatures. We also assumed that $\beta \ll 1$. Conclusion: the thermal agitation decreases the effective value of the B field in the plasma.

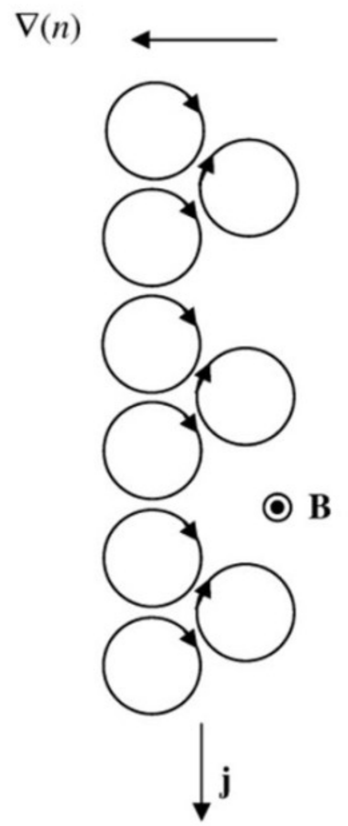
Magnetization current

The plasma particles carry a current equal to $\mathbf{j_{plasma}} = \nabla\times\mathbf{M}$

And we have just seen that $\mathbf{M} = -\frac{2nkT}{B_0^2}\mathbf{B}$ and we know that $\nabla\times\varphi\boldsymbol{B} = \nabla\varphi\times\mathbf{B}$

So there exist a current in the plasma if there is a gradient of the modulus of the magnetization vector,

$$\boldsymbol{J_M} = -\nabla\left(\frac{2nkT}{B_0^2}\right)\times\mathbf{B}$$



This is called the magnetization current.
(careful : « collective current »)

Motion of a particle in a magnetic and a constant electric field

We decompose the motion into parallel and perpendicular components

$$\begin{cases} \dot{v}_{\parallel} = qE_{\parallel}/m \\ \dot{\mathbf{v}}_{\perp} = \omega_c \mathbf{v}_{\perp} \times \mathbf{b} + q\mathbf{E}_{\perp}/m. \end{cases}$$

Along the perpendicular component, the particular solution of the equation is

$$\mathbf{v}_{\perp,p} \times \mathbf{b} = -\frac{q\mathbf{E}_{\perp}}{m\omega_c} \Rightarrow \mathbf{v}_{\perp,p} = \frac{\mathbf{E}_{\perp} \times \mathbf{b}}{B}.$$

The « cross-field drift »

Reminder: Galilean transform of the electric field $\mathbf{E}' = \mathbf{E} + \mathbf{u}_{R'/R} \times \mathbf{B}$,

(it is always possible to find a frame where the perpendicular electric field vanishes)

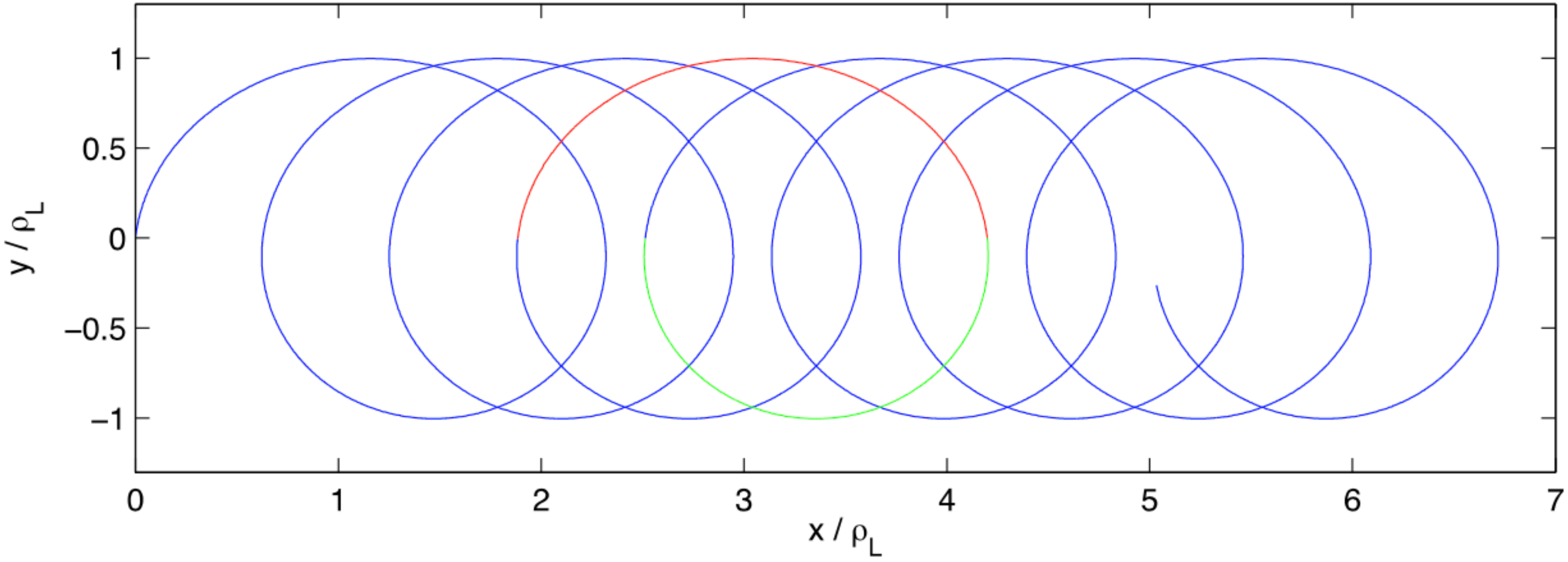
Motion of a particle in a magnetic and a constant force field

The same analysis applies, replacing E by F/q, so a perpendicular drift appears,

$$\mathbf{v}_F = \frac{\mathbf{F} \times \mathbf{B}}{qB^2}$$

Examples : gravitationnal drift and drifts arising from other inertial effects, like the curvature drift (cf. later)

The crossed field drift



- The particle starts at (0,0), assume the particle's charge is positive.
- What is the direction of the magnetic field ?
- Of the electric field ?
- Is it possible to know if the particle is an ion or an electron, from this trajectory only ?

Motion of a particle in a time varying electric field : resonance

In the plane perpendicular to the magnetic field, the equation of motion is $$\frac{dv_\perp}{dt} + i\omega_c v_\perp = \frac{q}{m} E_\perp,$$

Where all variables are complex. The solution of this equation is

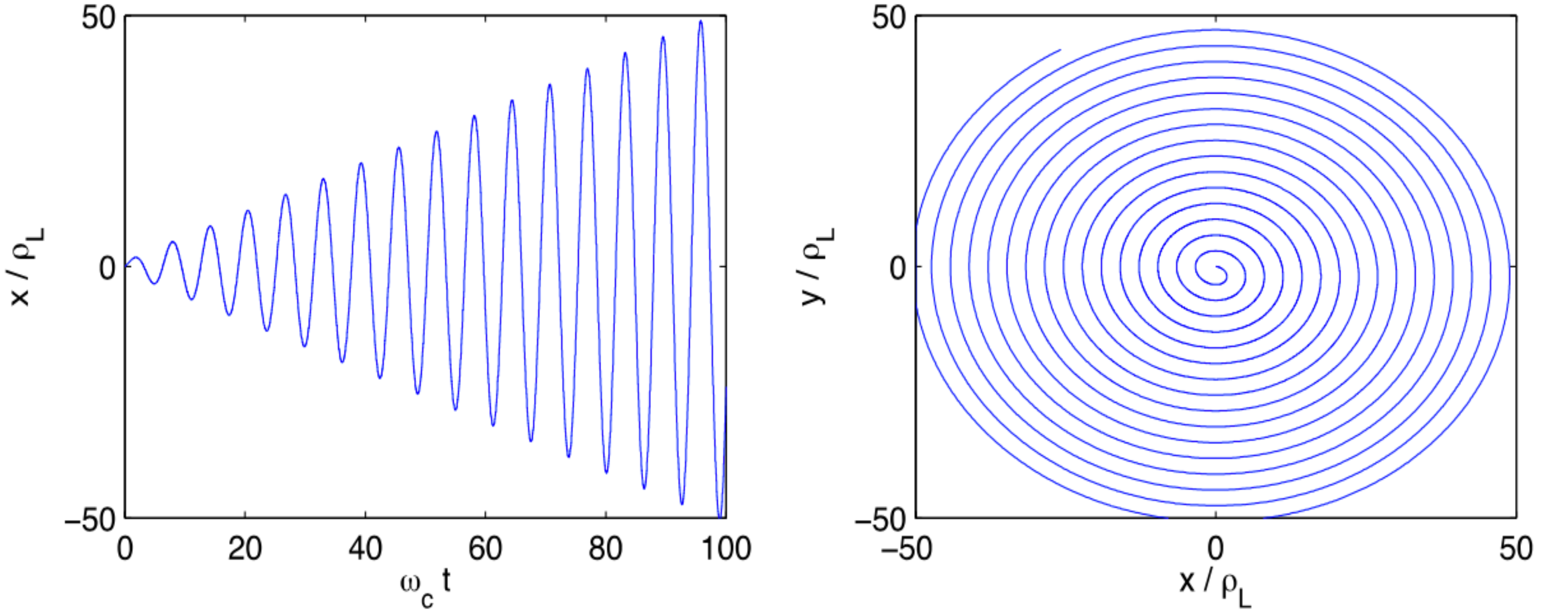
$$v_\perp(t) = e^{-i\omega_c t} \left(v_\perp(0) + \frac{q}{m} \int_0^t E_\perp(t') e^{i\omega_c t'} dt' \right)$$

Resonant behaviour :

$$E_\perp = E_0 \cos \omega_c \, t$$

The Larmor radius increases linearly with time

(wave-particle interaction)



Perpendicular drifts, guiding center theory

We consider variations of the fields very small on the scale of the particle gyroradius, and Taylor expand the field to first order. We get the equation of motion :

$$\frac{d\mathbf{V}_g}{dt} + \frac{d\mathbf{v}_\ell}{dt} = \frac{q}{m}\left(\mathbf{E}(\mathbf{R}_g) + \mathbf{r}_\ell \cdot \boldsymbol{\nabla}\mathbf{E}(\mathbf{R}_g) + (\mathbf{V}_g + \mathbf{v}_\ell) \times (\mathbf{B}(\mathbf{R}_g) + \mathbf{r}_\ell \cdot \boldsymbol{\nabla}\mathbf{B}(\mathbf{R}_g))\right).$$

In which we separated the guiding center motion from the fast cyclotron motion :

$$\mathbf{r}(t) = \mathbf{R}_g(t) + \mathbf{r}_\ell(t), \qquad \mathbf{v}(t) = \mathbf{V}_g(t) + \mathbf{v}_\ell(t),$$

With $< \boldsymbol{r}_\ell(t) > = 0$ and $< \boldsymbol{v}_\ell(t) > = 0$

Order 0 : the field are like homogeneous…

Keeping the 0th order terms only, we get, after averaging on time,

$$\mathbf{E}(\mathbf{R}_g) + \mathbf{V}_g^{(0)} \times \mathbf{B}(\mathbf{R}_g) = 0 \Rightarrow \mathbf{V}_g^{(0)} = \frac{\mathbf{E}(\mathbf{R}_g) \times \mathbf{B}(\mathbf{R}_g)}{B(\mathbf{R}_g)^2} = \mathbf{v}_\times(\mathbf{R}_g)$$

So the motion of the guiding center just consists in the cross field drift calculated with the field at the position of the guiding center

In the parallel direction, the motion is free from magnetic effect and consists in an acceleration in the given electric field.

In the following we assume that there is no parallel electric field.

Order 1 : effect of the field gradients

Keeping the 1th order terms and averaging on time, we get

$$\frac{d\mathbf{V}_g^{(0)}}{dt} = \frac{q}{m}\left(\mathbf{V}_g^{(1)} \times \mathbf{B}(\mathbf{R}_g) + \langle \mathbf{v}_\ell \times \mathbf{r}_\ell \cdot \boldsymbol{\nabla}\mathbf{B}(\mathbf{R}_g)\rangle\right).$$

And the average $\langle \mathbf{v}_\ell \times \mathbf{r}_\ell \cdot \boldsymbol{\nabla}\mathbf{B}(\mathbf{R}_g)\rangle = -\frac{v_\perp^2}{2\omega_c}\boldsymbol{\nabla}B = -\frac{\mu}{q}\boldsymbol{\nabla}B$

The acceleration of the guiding center can be decomposed in three terms :

$$\frac{d\mathbf{V}_g^{(0)}}{dt} = \frac{dv_\parallel}{dt}\mathbf{b} + v_\parallel\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{v}_\times}{dt}$$

The mirror force, and the conservation of μ

Projecting the equation of motion along the parallel direction, we get

$$m\frac{dv_{\|}}{dt} = -\mu \mathbf{b} \cdot \boldsymbol{\nabla} B$$

From which we demonstrate the the magnetic moment of the particle is a conserved quantity along the particle's trajectory. Indeed,

$$\frac{d}{dt}\left(\frac{1}{2}mv_{\|}^2\right) = -\mu v_{\|}\mathbf{b} \cdot \boldsymbol{\nabla} B = -\mu\frac{dB}{dt}$$

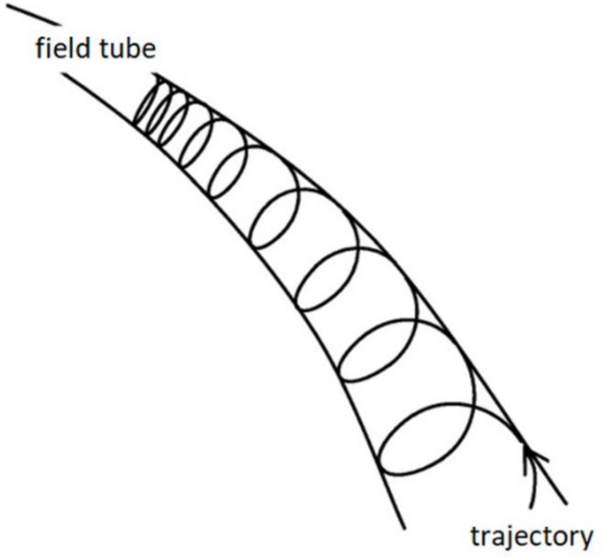
And the kinetic energy of the particle is conserved (no work from the magnetic field).

$$\frac{d}{dt}\left(\mathscr{E}_{\perp} + \mathscr{E}_{\|}\right) = \frac{d}{dt}(\mu B) - \mu\frac{dB}{dt} = B\frac{d\mu}{dt} = 0$$

Conservation of magnetic flux through the Larmor "current ring"

We have seen that the magnetic moment is conserved along the motion of a particle. A consequence, the magnetic flux through a surface resting on the contour defined by the cyclotron motion is also conserved. This can be seen from the expression of this flux

$$\Phi_B = \iint \mathbf{B} \cdot d\mathbf{S} \simeq B(s)\pi\rho(s)^2 = \frac{\pi m^2}{q^2}\frac{v_\perp^2}{B} = \frac{2\pi m}{q^2}\mu = const.$$



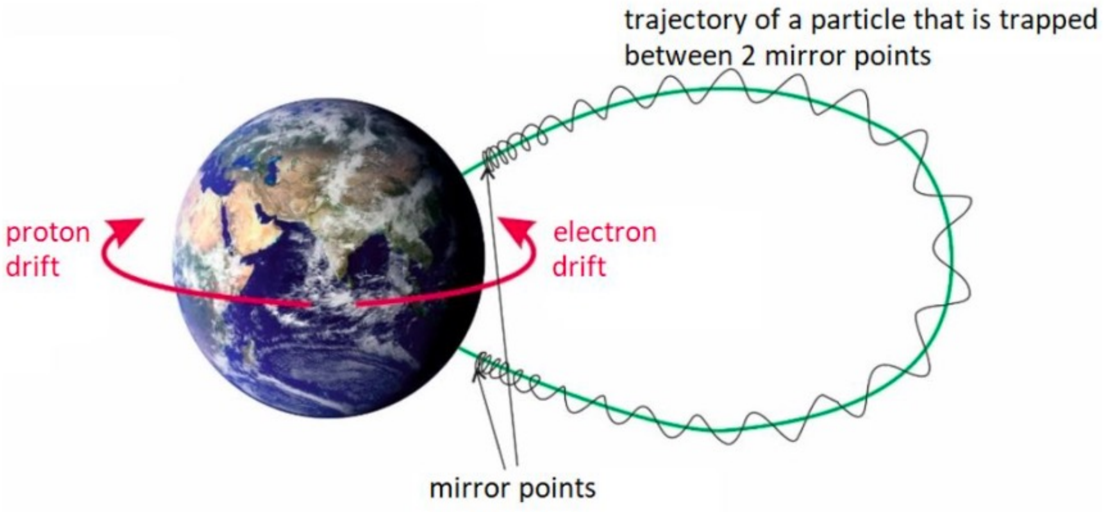
This provides an easy and intuitive way to visualize the trajectory of particles in complicated (but slowly varying) fields.

Mirror points

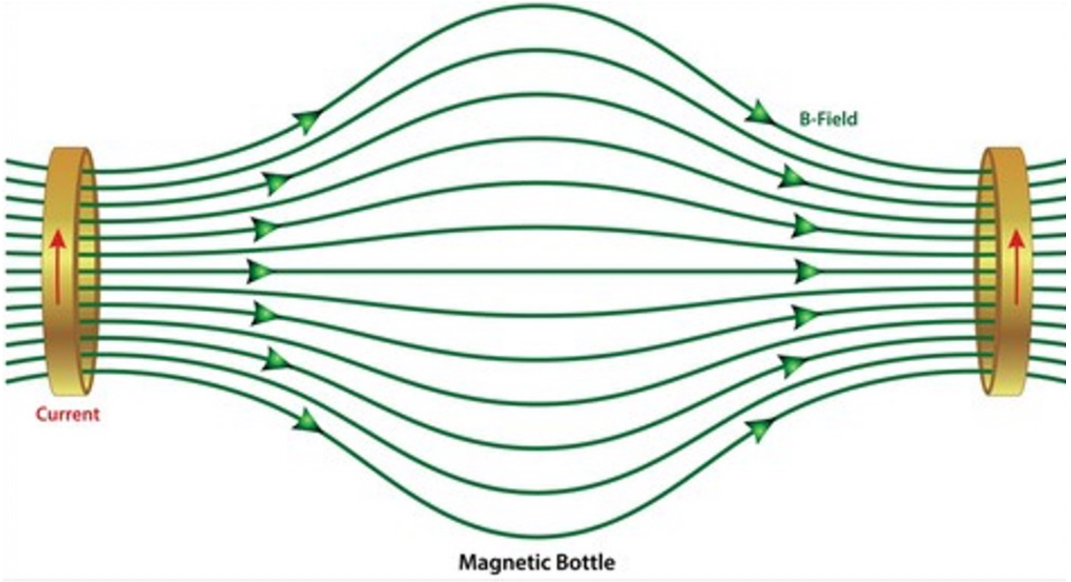
Because of the mirror force, the magnetic field modulus acts as a pseudo-potential for the motion of the particles along the field lines

$$\frac{1}{2}mv_{\parallel}^{2}(s) + \mu B(s) = \mathscr{E} = const.$$

The particle is reflected at positions where $B(s_m) = E/\mu$. These are called the mirror points of the particle.



Magnetic bottle, mirror ratio



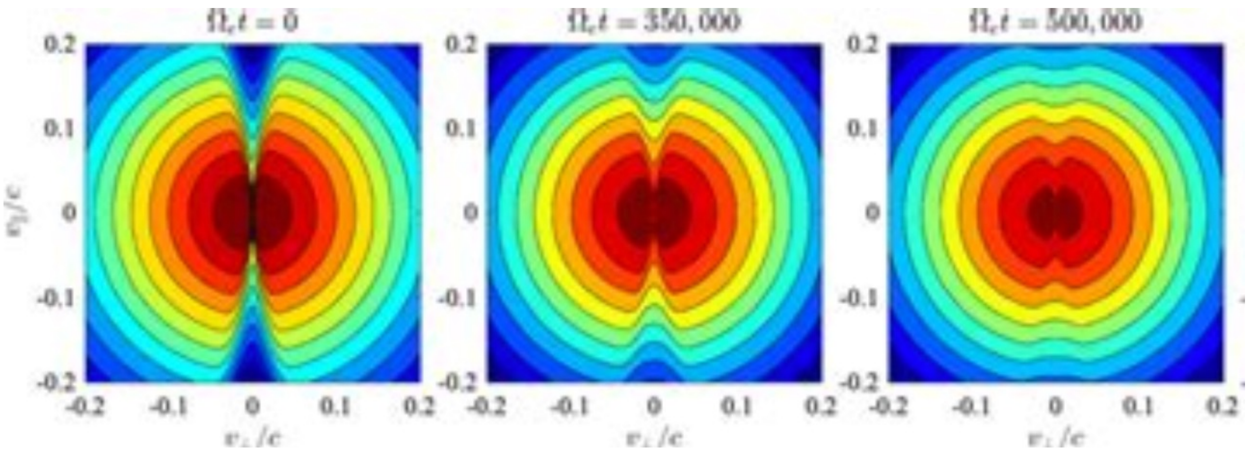
The mirror ratio R = Bmax / Bmin

Condition for escape : $\mathscr{E} > \mu B_{max}$

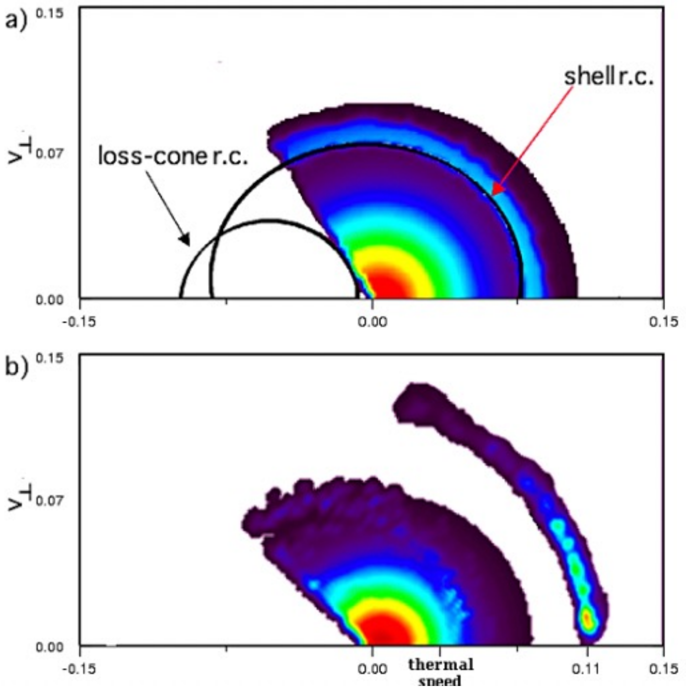
Can be rewritten as a function of the pitch-angle of the particle at the position where B=Bmin :

$$\sin^2 \theta_{min} < \frac{B_{min}}{B_{max}} = 1/R_m.$$

Loss cone



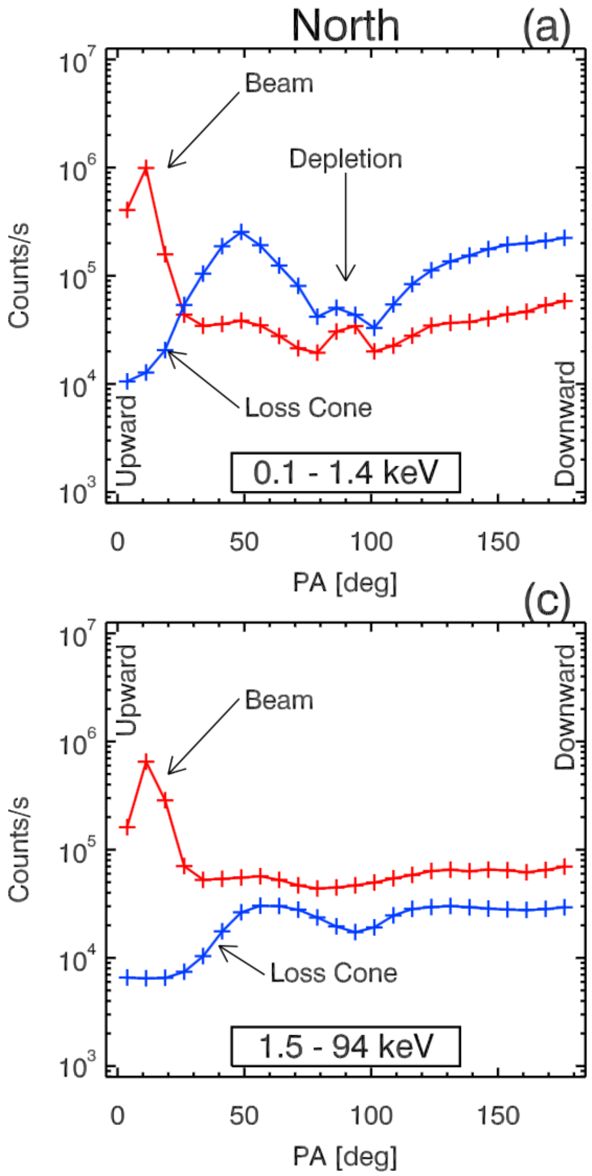
Loss cone and time evolution related to an electrostatic instability



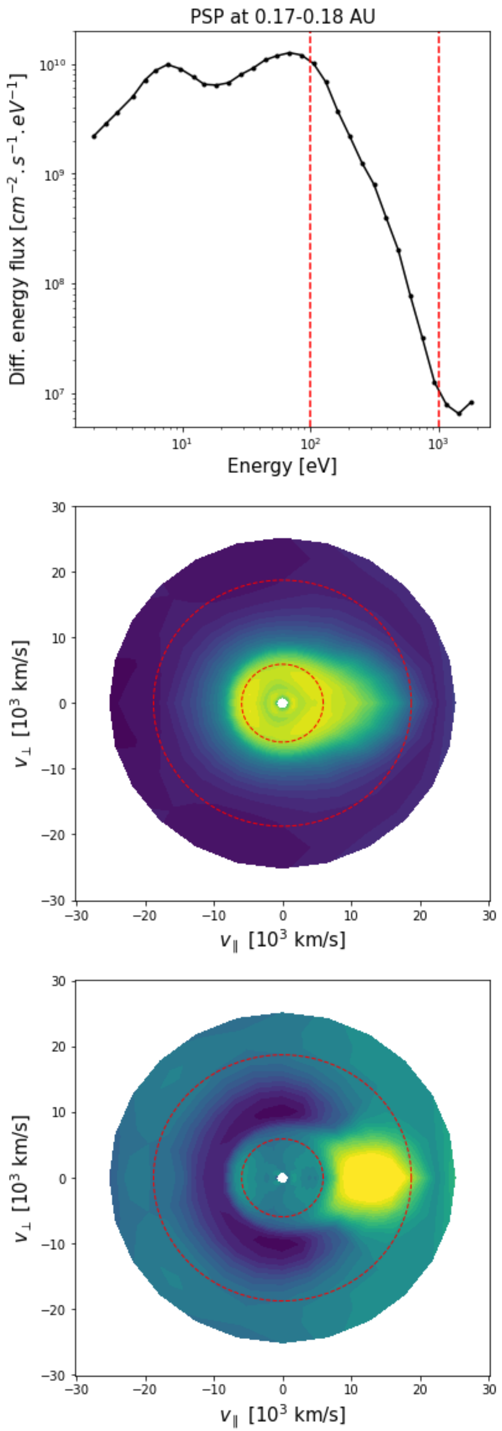
Hess et al, 2007

Loss cone in the Jovian environment

Electron distribution observations



Jupiter’s auroral regions



Solar wind at 0.17 AU

Perpendicular drifts

Going back to our first order equation, but now taking the vector product by **b**, we get the first order perpendicular motion of the guiding center

$$\mathbf{V}_g^{(1)} = \frac{1}{\omega_c}\mathbf{b} \times \left(\frac{\mu}{m}\boldsymbol{\nabla} B + v_\parallel \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{v}_\times}{dt}\right)$$

These three terms correspond to three types of perpendicular drifts

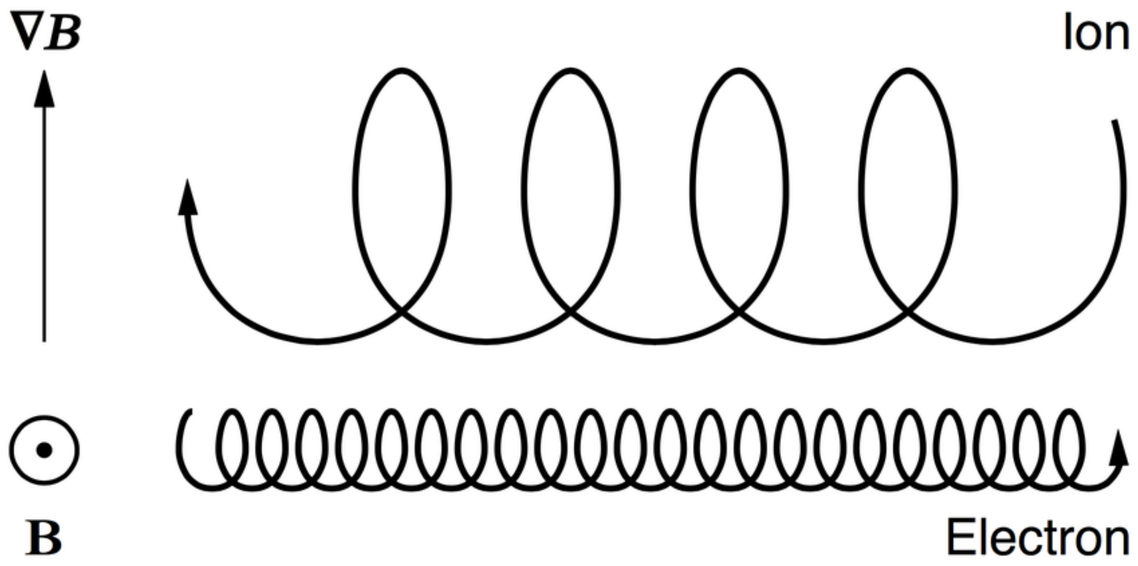
- The grad-B drift
- The curvature drift
- The polarization drift

- Note that the two last terms actually arise from inertial effects due to the acceleration of the guiding center at order 0. Also note that these drifts depend on the sign of the particle's charge, and give rise to associated currents.

Grad-B drift

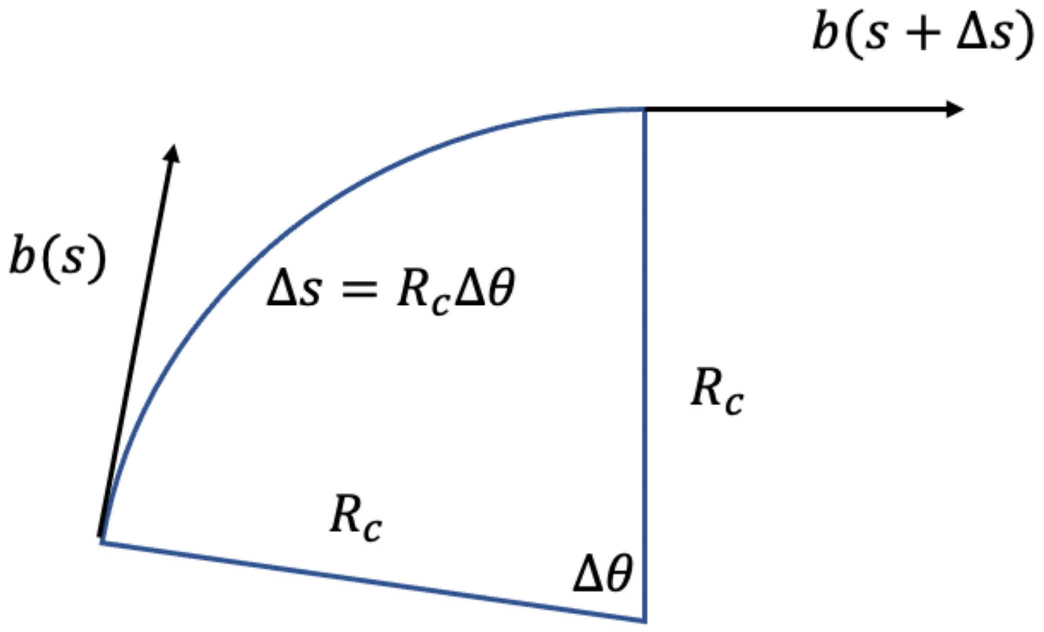
The grad-B drift is the perpendicular counterpart from the mirror force seen previously. Its expression is

$$\mathbf{v}_\nabla = \frac{\mu}{q}\frac{\mathbf{B}\times\nabla B}{B^2} = -\frac{mv_\perp^2}{2qB}\frac{\nabla B\times\mathbf{B}}{B^2}$$



Curvature drift

The curvature drift is due to the change of direction of the velocity of the particle with time when the field line is curved. Considering the osculating circle at the position of the particle.



We have $$\frac{d\boldsymbol{b}}{dt} \simeq v_\parallel \frac{d\boldsymbol{b}}{ds}$$

$$\boldsymbol{b}(s+\Delta s) = \mathrm{R}(\Delta\theta)\boldsymbol{b}(s) = \boldsymbol{b}(s) - \sin\Delta\theta\, \boldsymbol{n}$$

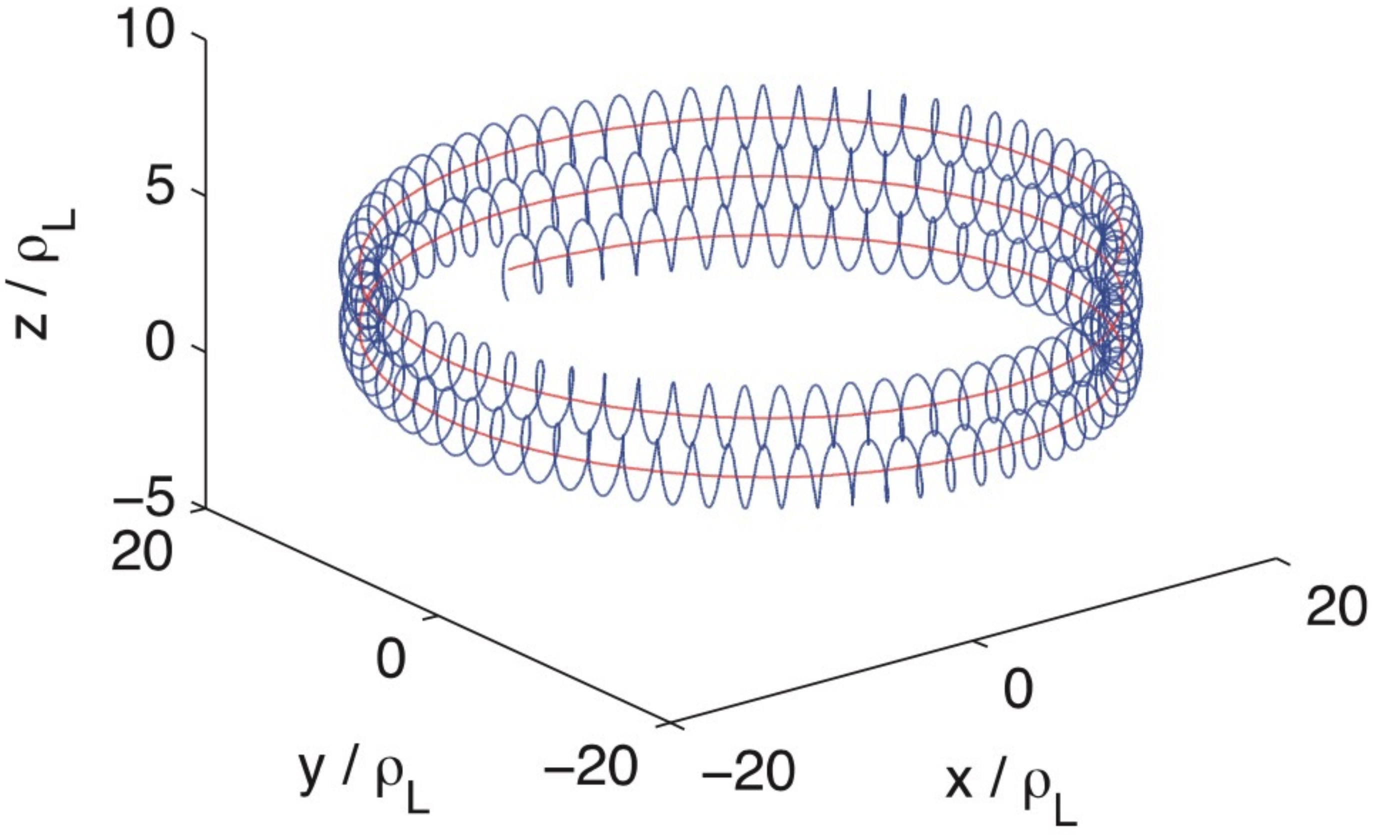
So that

$$\frac{d\boldsymbol{b}}{ds} = \lim_{\Delta\theta\to 0} -\sin\frac{\Delta\theta}{\Delta s}\boldsymbol{n} = -\frac{\boldsymbol{n}}{R_c}$$

Curvature drift

$$\mathbf{v}_c = \frac{m}{q}\frac{\mathbf{B}\times v_\parallel^2(\mathbf{b}\cdot\nabla\mathbf{b})}{B^2} = \frac{mv_\parallel^2}{qR_c}\frac{\mathbf{n}\times\mathbf{B}}{B^2}$$

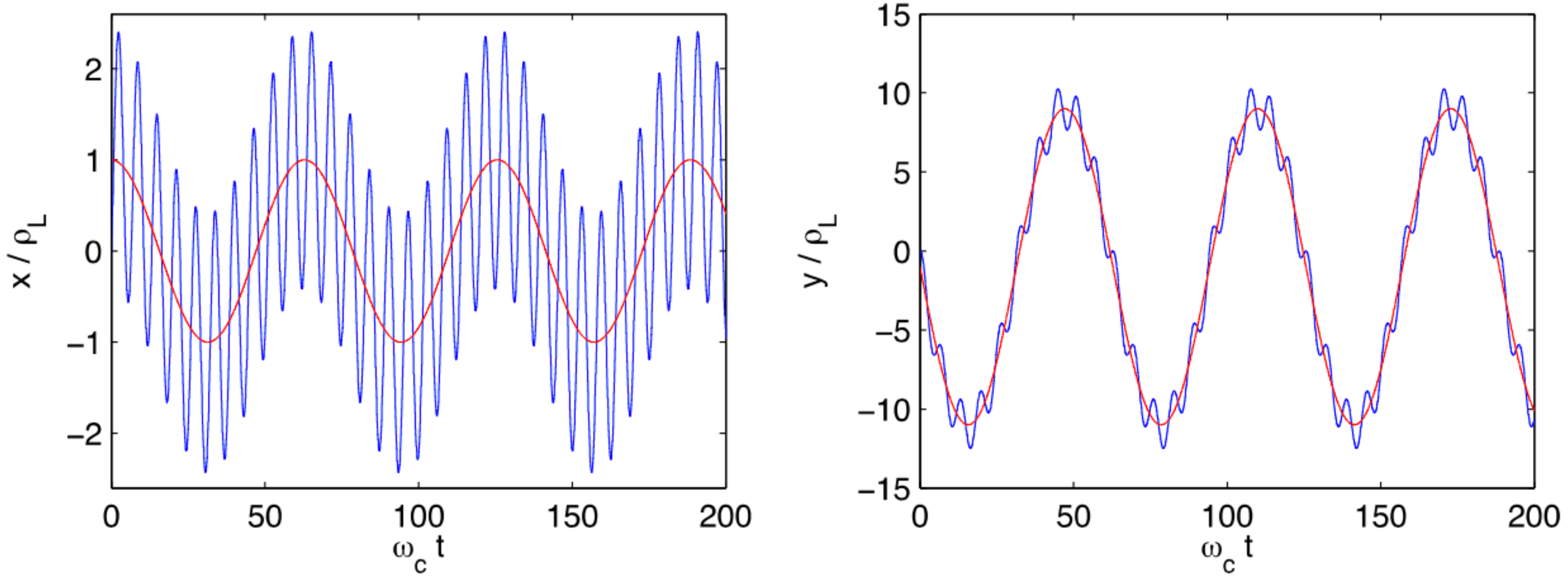
(is it an ion or an electron on the figure ?)



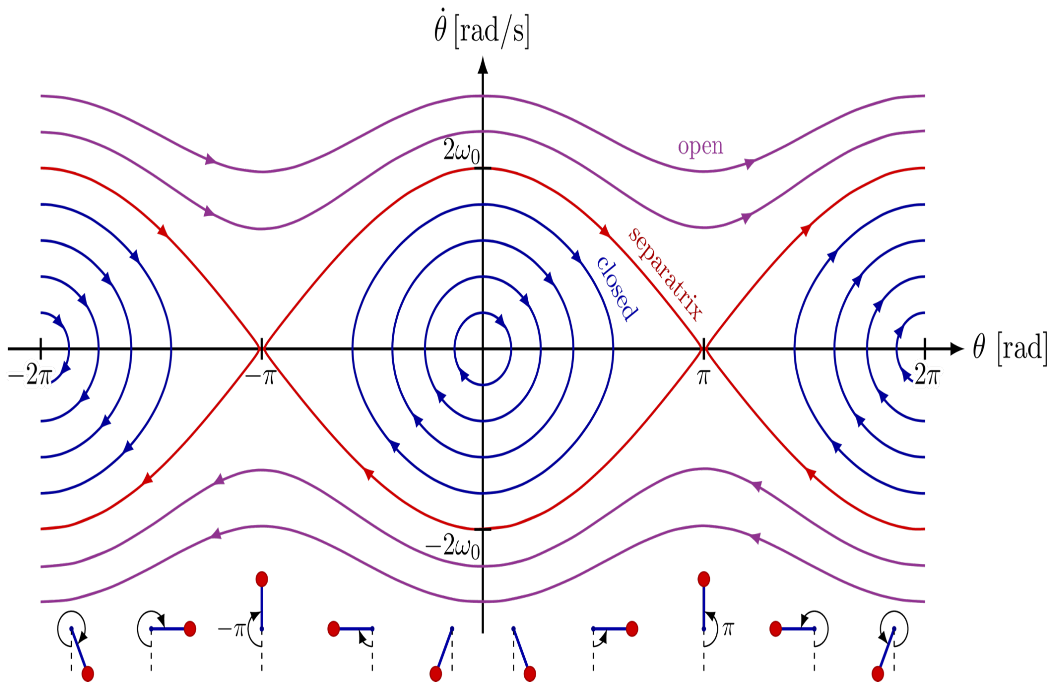
The polarization drift

The last term of the equation describes inertial effects due to the change in the cross field drift velocity. Considering only the change due to a time variation of the electric field, we obtain the expression of the polarization drift

$$\mathbf{v}_p = \frac{m}{qB^2}\frac{\mathbf{B}\times(\dot{\mathbf{E}}_\perp \times \mathbf{B})}{B^2} = \frac{m}{qB^2}\frac{d\mathbf{E}_\perp}{dt},$$



The adiabatic invariants



The surface embedded by periodic phase-space trajectories is conserved if the parameters defining the trajectory are slowly varying in time (slow=compared to the oscillation period)

$$I = \oint \mathbf{p} \cdot d\mathbf{q}$$

One can also show that

$$I = \int_{T_q} W_q(t)dt = \langle W_q(t) \rangle T_q,$$

The first adiabatic invariant

The first, and fastest, periodic motion to be identified for a particle in a magnetic field is the cyclotron motion. Applying the previous relation, we get

$$I_1 = \langle W_q(t) \rangle T_q = \frac{1}{2} m v_\perp^2 \frac{2\pi}{\omega_c} = \frac{2\pi m}{q} \mu$$

So the first adiabatic invariant is nothing but the magnetic moment of the particle, which is conserved under slow time variations of the magnetic field.

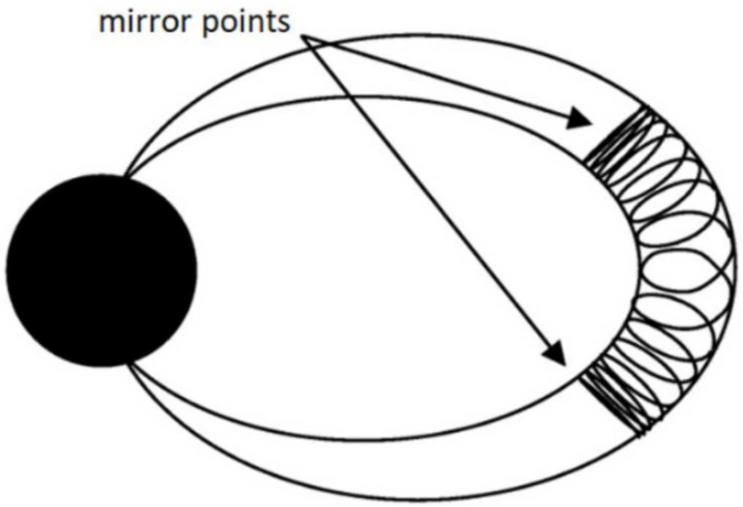
This is used for particle’s acceleration in « betatron » devices.

The second adiabatic invariant

Consider the motion of a particle along a field line along which the modulus of the magnetic field varies. The particle will oscillate between its mirror points. This oscillatory motion is certainly of much longer period than the cyclotron motion.

If the parameters defining the field line are slowly varied in time (compared to the bounce period), then

$$I_2 = \langle W_s(t) \rangle T_s = \frac{1}{2} m \langle v_\parallel^2 \rangle T_b$$



The third adiabatic invariant

The third adiabatic invariant arises when considering the bounce motion as a fast motion (the particle can then be viewed as a « magnetic shell », i.e. the flux tube bounded by the two mirror points).

The particle shell will in general rotate under the effect of magnetic drifts (examples of a mirror trap, or of the earth dipole).

The invariant is better expressed using the integral on particle's generalized momentum

$$I_3 = \int_0^{2\pi} p_\phi r d\phi \simeq q \oint \mathbf{A} \cdot d\boldsymbol{\ell} = q\Phi_B \sim q\pi R^2 B_0$$

Triple periodicity of the motion in the Earth's magnetic dipole